INTEREST THEORY

FINANCIAL MATHEMATICS AND DETERMINISTIC VALUATION

SECOND EDITION

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Interest Theory

Financial Mathematics
and
Deterministic Valuation

Second Edition

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Table of Contents

Introduction

This textbook strives to move you through the material quickly, highlighting important concepts along the way. The sooner you understand the material, the sooner you can begin the important work of practicing questions. The key to mastering this material is to work as many questions as possible.

Intermediate calculations in this text are not rounded. As you work through the examples and questions, you will match the solutions if you keep the numbers stored in your calculator.

We've formatted this textbook so that our comments, warnings, tips, examples, and Key Concepts are readily recognized.

An italicized section is a side comment from the author.

The friendly owl at left provides helpful comments, often adding context to the surrounding material.

The owl with a book gets into the details. It isn't always necessary to delve into this level of detail, so this icon indicates material that is optional during the first reading of the material.

The owl with an exclamation point sign warns of potential pitfalls or traps. We've identified certain mistakes as being particularly easy to make, and we use this icon to warn you away from them.

The owl with a magic wand alerts you to tips and tricks that can save you time while learning the material and/or working the questions.

Example 0.00

Examples are denoted with double lines in the left margin. The word "Example" appears to the left of the double lines. An example begins with a question.

Solution

An example concludes with a solution. The word "Solution" appears to the left of the double lines.

0.00

Key Concepts

Key Concepts are denoted with a solid dark line in the left margin and a key icon. It is important to understand the Key Concepts thoroughly. In most cases, the Key Concepts should be memorized.

If you encounter any errors in this text, please let us know. Send any questions or comments to Info@ActuarialBrew.com.

Full solutions to the practice questions at the end of each chapter can be found online at www.ActuarialBrew.com. Each solution has the following key to indicate the question's degree of difficulty. The more boxes that are filled in, the more difficult the question:

Easy: ■□□□□

Very Difficult: ■■■■■

Chapter 1: Setting the Stage

This chapter introduces basic terminology and lays the groundwork for the subsequent chapters.

1.01 BA II Plus Calculator

This text frequently references the Texas Instruments BA II Plus calculator. The instructions for the BA II Plus are the same as those for the BA II Plus Professional, and either calculator is an excellent choice for use with this text.

Digits Displayed

Although the BA II Plus retains 13 digits of accuracy, it only shows 2 decimal places when at its default setting. The following keystrokes change the calculator to show up to 9 decimal places:

[2nd] [FORMAT] 9 [ENTER] [2nd] [QUIT]

Order of Operations

The calculator uses the chain calculation method when at its default setting. The chain calculation executes the operations as they are entered, so we have:

Chain: $4+2\times 5=30$

The calculation method can be changed to the algebraic operating system (AOS), which uses the standard order of operations:

[2nd] [FORMAT] ↓↓↓↓ [2nd] [SET] [2nd] [QUIT]

Now we have:

AOS: $4+2\times 5=14$

Instructions in this text are based on the AOS calculation method.

Payments per Year and Compounding Periods per Year

The BA II Plus has internal settings for payments per year (P/Y) and compounding periods per year (C/Y). Some BA II Plus calculators default these settings to 12. Changing the P/Y setting to 1 will automatically change the C/Y setting to 1:

[2nd] [P/Y] 1 [ENTER] [2nd] [QUIT]

Instructions in this text are based on both P/Y and C/Y being set equal to 1.

If you take an actuarial exam, the exam administrator will reset your calculator prior to the exam. This means that it will then revert to using the chain calculation method, and it may change the settings for P/Y and C/Y. After it has been reset, you can easily change it back to the settings shown above. If you would like to practice, you can reset the calculator as follows:

[2nd] [RESET] [ENTER] [2nd] [QUIT]

Clearing the Worksheets

Instructions in this text are based on an assumption that that the relevant worksheet's register has been cleared. You can clear the time value of money (TVM) worksheet as follows:

[2nd] [CLR TVM]

To clear any of the other worksheets, you must be in the worksheet. The command for clearing a non-TVM worksheet is:

[2nd] [CLR WORK]

Beginning Vs. End of Period

Unless stated otherwise, instructions in this text are based on an assumption that the calculator is set to treat payments as occurring at the end of each period.

To toggle the calculator between treating payments as occurring at the beginning of each period or the end of each period, use the following key strokes:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

When the calculator is set to treat payments as occurring at the beginning of each period, the letters BGN appear in the upper right of the display.

Unless stated otherwise, the instructions in this text are based on an assumption that BGN does not appear in the upper right of the display.

Basic Equation in TVM Worksheet

The BA II Plus allows us to solve for any one of the following 5 variables if given the other 4: *N*, *I/Y*, *PV*, *PMT*, and *FV*. These variables are components of the two equivalent equations below:

$$PV + PMT \times a_{\overline{N|}I/Y} + FV \times v^N = 0$$

$$PV \times (1 + I/Y)^N + PMT \times s_{\overline{N|}I/Y} + FV = 0$$

We do not use the expression above until we get to Section 6.02, but it is included here so that we have the useful information about the calculator in one place. Don't worry if the variables and the equations don't make sense to you yet!

Rounding

Although this text uses rounded values when showing intermediate steps, the full values are retained for calculation purposes. For example, the conventions of this text permit the expression on the left but not the one on the right:

YES	NO
$\frac{2}{3} \times 2 = 0.67 \times 2 = 1.33$	$\frac{2}{3} \times 2 = 0.67 \times 2 = 1.34$

1.02 Valuation Date and Payment Date

Valuing assets can be difficult. As anyone who has ever watched *Antiques Road Show* on PBS can attest, the value of an asset isn't always obvious. But valuing cash seems like a much simpler task. What is the value of $100? It's obviously $100.

Valuing cash is a simple proposition when the valuation date and the payment date are the same. Below are some simple examples where the valuation date is equal to the payment date:

- If someone gave you $50 last month, then it was worth $50 at that time.
- The value now of a payment of $100 now is $100.
- The value in one year of a payment of $200 at that time is then $200.

When the payment date does not match the valuation date, however, valuation can become more complicated.

What is the value now of $100 to be received in one year? As a starting point, observe that receiving $100 in one year is less desirable than receiving $100 now. This is because if we have $100 now, then we can deposit that $100 in a bank account and the balance will grow to more than $100 in one year. The difference between the $100 deposited and the bank balance in one year is known as **interest**.

The **time value of money** is the value at a particular point in time of a payment or set of payments. As noted above, a payment of $100 to be made one year from now has a value of less than $100 now, and its value in one year is $100. The time value of money is also known as the **current value** at a particular point in time.

Recognizing the time value of money, we can set up an **equation of value** for a payment or set of payments that corresponds to a particular point in time. For example, suppose that \$100 deposited in a bank account now will grow to \$103 at the end of one year. In that case, an equation of value for time 0 would tell us that the **present value** today of the \$103 is \$100:

Time 0 equation of value: $PV_0 = 100$

where:

$$PV_t = \begin{array}{c}\text{Present value at time } t \text{ of a payment} \\ \text{or payments occuring on or after time } t\end{array}$$

The present value of a future payment is the amount that must be lent in order to receive that payment later. In subsequent chapters, we see that the present value function depends on the type of interest rate used.

We can also set up an equation of value for time 1 that tells us that the **future value**, or **accumulated value**, in one year of \$100 payable now is \$103:

Time 1 equation of value: $AV_1 = 103$

where:

$$AV_t = \begin{array}{c}\text{Accumulated value at time } t \text{ of a payment} \\ \text{or payments occuring on or before time } t\end{array}$$

The accumulated value of a payment is the amount that a lender receives for lending that payment earlier. In subsequent chapters, we see that the accumulated value function depends on the type of interest rate used.

*When the accumulated value is expressed as a function of the time elapsed, the function is sometimes referred to as the **amount function**.*

1.03 Current Values

The current value of a set of payments is the sum of the accumulated value of the payments that occur before the valuation date, the value of any payment made on the valuation date, and the present value of the payments made after the valuation date:

$$CV_t = AV_t(\text{payments occurring before } t) + Pmt_t + PV_t(\text{payments occurring after } t)$$

where:

$CV_t =$ Current value at time t of a set of payments

$Pmt_t =$ Payment made at time t

If an entire set of payments is made on or before the valuation date t, then the time-t current value is an accumulated value. If an entire set of payments is made on or after the valuation date t, then the time-t current value is a present value. Although accumulated values and present values are current values, the converse is not necessarily the case.

1.04 Loans Repaid with a Single Payment

Consider a transaction between two parties, in which the first party receives an amount of money now from the second party in exchange for agreeing to make a future, larger payment to the second party at some point in the future. This transaction is a **loan**.

The amount of money exchanged now is called the **principal**. The amount by which the future payment exceeds the principal is called the **interest**:

Future Payment $-$ Principal $=$ Interest

The party that accepts the principal now is the **borrower**. The borrower agrees to repay the principal plus interest in the future. The present value of the loan is the principal:

Present Value $=$ Principal

The party that provides the principal now is the **lender**. The lender receives the principal plus interest in the future. The accumulated value of the loan is the future payment:

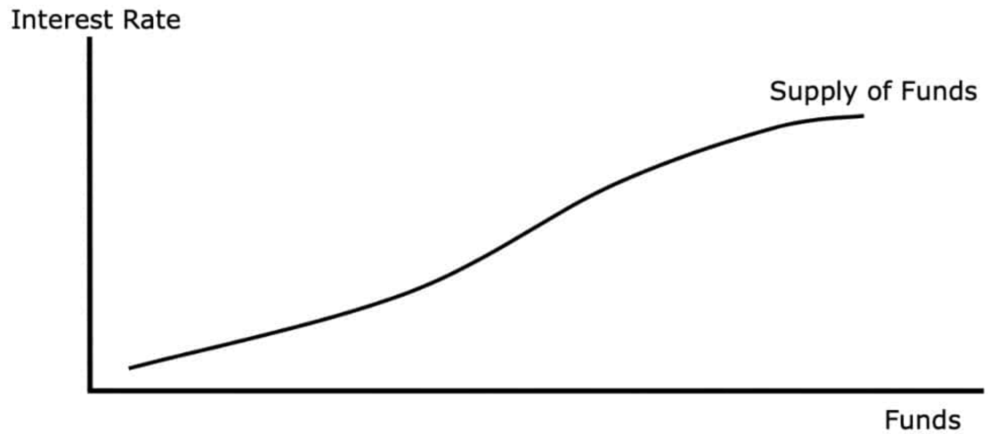
$$\text{Accumulated Value} = \text{Future Payment}$$

The accumulated value is also known as the future value.

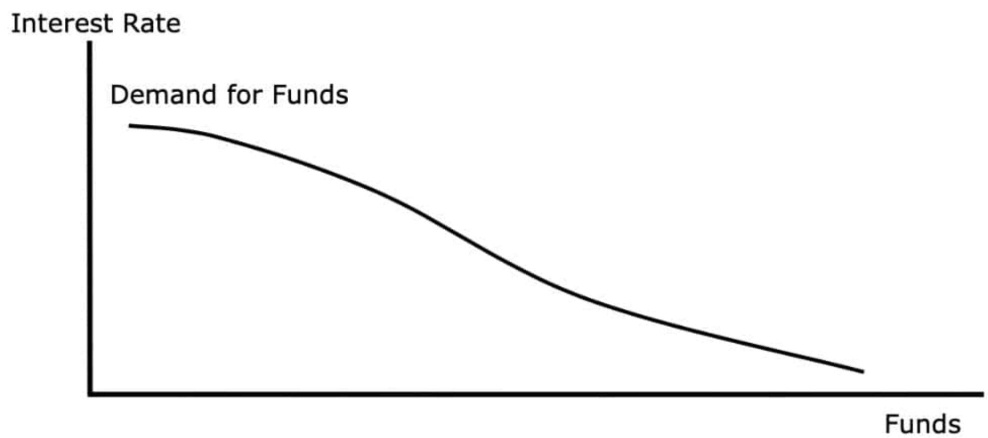
1.05 Supply and Demand Curves for Funds

An interest rate is used to calculate the amount of interest that is paid on a loan. There are many different ways of expressing interest rates, and this text explores those ways in the upcoming chapters. For now, however, it is sufficient to know that higher interest rates lead to higher interest payments.

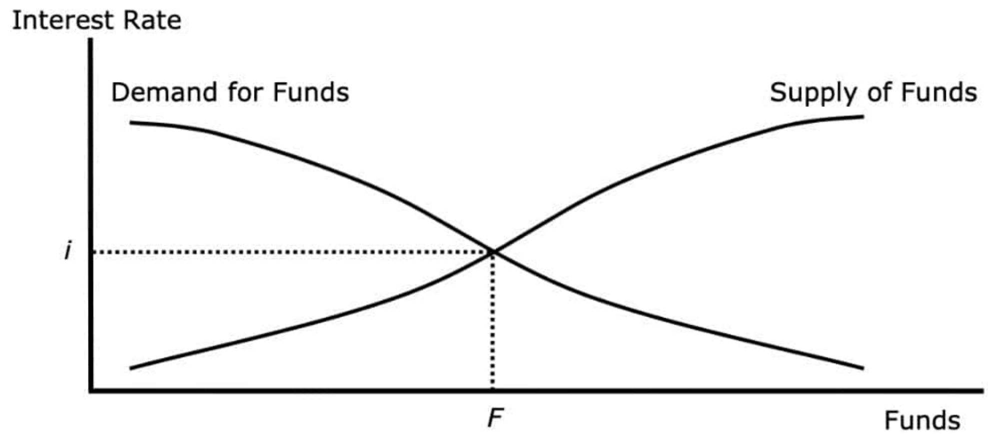
A lender is motivated by the opportunity to earn interest. In order to earn interest, the lender must forgo current consumption because the lender is unable to use the loaned funds to consume goods and services until the loan is repaid. Interest is therefore compensation for deferred consumption. A higher interest rate encourages lenders to lend more. If the interest rate is relatively low, then lenders makes fewer loans. The quantity of funds made available by lenders increases with the interest rate, as shown below in the graph of a supply curve for funds.



A borrower has the opposite response to interest rates. A borrower wishes to consume now and is willing to pay a higher price in the future in order to consume immediately. Interest, from the borrower's perspective, is the cost to consume immediately when resources are not currently available. The higher the interest rate, the higher the future price of immediate consumption because the future price is the principal plus interest. Therefore, a higher interest rate causes the borrower to borrow less, while a lower interest rate causes the borrower to borrow more. This can be observed in the graph below of a demand curve for funds.



The equilibrium interest rate, *i*, can be observed where the supply and demand curves intersect.



Suppose the interest rate is less than *i*. In that case, the quantity of funds demanded is greater than *F* and the quantity of funds supplied is less than *F*. Some borrowers are therefore unable to obtain loans. These borrowers contact potential lenders and offer to pay more than *i*, thereby enticing lenders to make more funds available. This continues until the interest rate increases to *i*.

If the interest rate is greater than *i*, then the quantity of funds supplied is greater than *F* and the quantity of funds demanded is less than *F*. Some lenders are unable to place their funds with a borrower. These lenders contact borrowers and offer to lend at less than *i*, thereby enticing borrowers to take out more loans. This continues until the interest rate falls to *i*.

The discussion above describes how the supply and demand for loanable funds determines the interest rate, but to fully understand the interest rates observed in the real world, we must take many other considerations into account, including the following:

- Term – the length of time until the loan matures
- Credit quality – the likelihood that the borrower will make the required payments
- Forecasts – the predicted levels of future interest rates and other macroeconomic variables
- Liquidity – the ability of a lender to sell a loan with little to no transaction costs
- Inflation – the loss of purchasing power of a currency over time

Furthermore, decisions regarding saving versus lending can be affected by lifestyle decisions and demographics. For example, a large number of young adults may choose to borrow in order to buy houses. Conversely, a large number of older adults may choose to increase their savings as they anticipate retirement. As a result, the supply and demand curves shift and change shape over time.

1.06 Day Count Convention

Unless otherwise specified, this text is based on the following assumptions:

- There are 30 days in a month.
- There 365 days in a year.
- There are 52 weeks in a year.
- There are 12 months in a year.

Only the final assumption is indisputable. In fact, the other assumptions aren't even internally consistent. They suffice, however, when only one assumption is required.

Example 1.01 Ann lends money for 5 days. For how many years does Ann lend the money?

Solution The number of years that Ann lends the money is:

$$\frac{5}{365} = \mathbf{0.0137}$$

Example 1.02 Ann lends money for 5 days. For how many months does Ann lend the money?

Solution The number of months that Ann lends the money is:

$$\frac{5}{30} = \mathbf{0.1667}$$

Example 1.03 Ann lends money for 2 weeks. For how many years does Ann lend the money?

Solution The number of years that Ann lends the money is:

$$\frac{2}{52} = \mathbf{0.0385}$$

In Section 13.08, we introduce some other ways of counting the number of days in an interval of time.

1.07 Useful Formulas

The formulas below are provided for convenient reference.

In the following formulas for derivatives, u and v are functions of x, and a is a constant:

Product rule: $(uv)' = u'v + uv'$

Quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Exponential rule: $\left(a^x\right)' = a^x \ln(a)$

Logarithm rule: $\left(\ln(x)\right)' = \frac{1}{x}$

The sum of a geometric series is the first term of the series minus the term that would come next, divided by 1 minus the factor:

$$\sum_{t=0}^{n-1} ar^t = \frac{a - ar^n}{1 - r}$$

If a quadratic equation does not factor easily, then the quadratic formula can be useful:

$$ax^2 + bx + c = 0 \qquad \Rightarrow \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.08 Questions

Question 1.01

Consider the following statements:

I. The future value of a set of payments is also known as the accumulated value of the payments.

II. The current value of a set of payments is always the present value of payments that will occur in the future.

III. When a loan is to be repaid by a single payment in the future, the interest is equal to the difference between that payment and the principal.

IV. A lender pays interest to receive funds now.

Which of the statements is TRUE?

A I only B II only C I and III only D II and IV only E IV only

Question 1.02

Which of the statements regarding the market for loanable funds is FALSE?

A Interest compensates the lender for deferring consumption.

B The supply of funds available for lending increases with the interest rate.

C The demand for funds to borrow decreases with the interest rate.

D Interest is the cost that a borrower pays to consume immediately when resources are not currently available.

E Supply and demand curves do not change over time.

Chapter 2: Simple Interest and Discount

2.01 Simple Interest Rates

Simple interest rates can be used to compute the interest for a very simple transaction. If the simple interest rate is i per unit of time, then the interest per unit of time is the principal times the interest rate:

$$\text{Interest per unit of time} = \text{Principal} \times i$$

We often use one year as our unit of time, in which case i is an annual interest rate.

Using simple interest, the amount of interest is calculated by multiplying the simple interest rate times the principal times the amount of time that the loan is in effect. If the loan lasts for t units of time then:

$$\text{Interest} = \text{Principal} \times i \times t$$

Unless otherwise specified, we assume that the unit of time is one year.

Example 2.01

A loan of $1,000 is made now. The simple interest rate is 5% per year. At the end of 5 years, what is the amount of interest earned over the 5 years?

Solution

The amount of interest is:

$$\text{Interest} = \text{Principal} \times i \times t = 1{,}000 \times 0.05 \times 5 = \mathbf{250}$$

The present value of the loan is equal to the principal:

$$\text{Present Value} = \text{Principal}$$

The accumulated value is equal to the present value plus the interest:

$$\begin{aligned}\text{Accumulated Value} &= \text{Present Value} + \text{Interest}\\ &= \text{Present Value} + (\text{Present Value}) \times i \times t\\ &= (\text{Present Value})(1 + it)\end{aligned}$$

The ratio of the accumulated value to the present value is known as the **accumulation function**, and it is the accumulated value of an initial investment of 1. It is denoted by $a(t)$, and under simple interest, the accumulation function is:

$$a(t) = \frac{\text{Accumulated Value}}{\text{Present Value}} = 1 + it$$

2.01

Present Value and Accumulated Value Under Simple Interest

The accumulated value of a payment is the present value of the payment times the accumulation function:

$$AV_t = PV_0(1 + it)$$

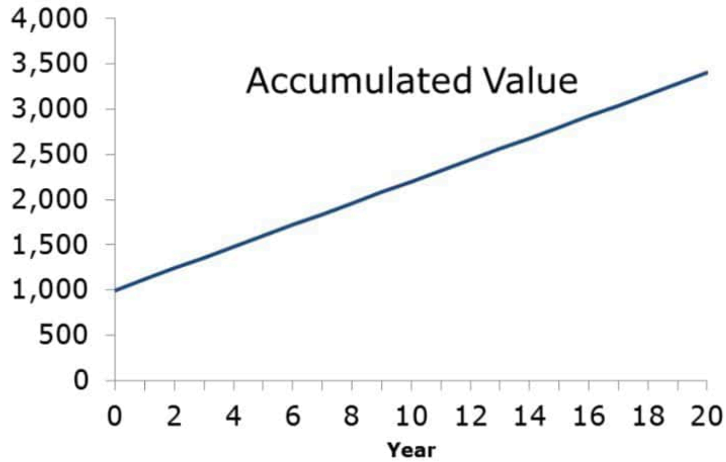
where:

AV_t = Accumulated value at time t

PV_0 = Present value at time 0

i = Simple interest rate per unit of time

The graph below uses a simple interest rate of 12% to show how $1,000 accumulates over 20 years. Under simple interest, the accumulated value is a straight line.



We can rearrange the formula above to find the present value of an accumulated value:

$$AV_t = PV_0(1 + it)$$

$$PV_0 = \frac{AV_t}{(1 + it)}$$

Example 2.02 The simple interest rate is 4% per year. How much money would Jenny need to deposit now to have $12,870 in 8 years?

Solution The amount to be deposited now is the present value of $12,870:

$$PV_0 = \frac{AV_8}{(1 + 8i)} = \frac{12,870}{1 + 8 \times 0.04} = \frac{12,870}{1.32} = \mathbf{9,750}$$

The next example verifies the result from the preceding example.

Example 2.03 The simple interest rate is 4% per year. If Jenny deposits $9,750 now, what is her accumulated value at the end of 8 years?

Solution The accumulated value is:

$$AV_8 = PV_0(1 + 8i) = 9,750(1 + 8 \times 0.04) = 9,750(1.32) = \mathbf{12,870}$$

If the time interval is not an integer, then interest is credited for the partial unit of time.

Example 2.04 The simple interest rate is 4% per year. If Jenny deposits $10,000 now, what is her accumulated value at the end of 18 months?

Solution Since the simple interest rate is expressed as an annual value, we use years as our unit of time. 18 months is equal to 1.5 years, and her accumulated value at the end of 1.5 years is:

$$AV_{1.5} = PV_0(1 + 1.5i) = 10,000(1 + 1.5 \times 0.04) = 10,000(1.06) = \mathbf{10,600}$$

2.02 Simple Discount Rates

As we saw above, a simple interest rate is multiplied times the principal to obtain the amount of interest per unit of time. A simple discount rate, on the other hand, is multiplied times the accumulated value to obtain the interest per unit of time, which in this context is often called the discount per unit of time:

If the simple discount rate is d per unit of time, then the discount per unit of time is:

$$\text{Discount per unit of time} = (\text{Accumulated value}) \times d$$

Using simple discount, the amount of discount is calculated by multiplying the simple discount rate times the accumulated value times the amount of time that the loan is in effect. If the loan lasts for t units of time then:

$$\text{Discount} = \text{Interest} = (\text{Accumulated Value}) \times d \times t$$

Under both simple interest and simple discount, the accumulated value is the present value plus the interest:

$$\text{Accumulated Value} = \text{Present Value} + \text{Interest}$$

$$\text{Accumulated Value} = \text{Present Value} + \text{Discount}$$

Rearranging the final expression above, the present value is equal to the accumulated value minus the discount:

$$\text{Present Value} = \text{Accumulated Value} - \text{Discount}$$

$$\text{Present Value} = \text{Accumulated Value} - (\text{Accumulated Value}) \times d \times t$$

$$\text{Present Value} = (\text{Accumulated Value})(1 - dt)$$

If t becomes too large, then the present value becomes negative. Therefore, simple discount only makes sense when t is less than the inverse of d:

$$t < \frac{1}{d}$$

Example 2.05

The simple discount rate is 6% per year. Determine the present value of $1,000 to be paid in:

a. 1 year

b. 10 years

c. 20 years

Solution

The present value is the accumulated value minus the discount:

a. $PV_0 = 1{,}000(1 - 0.06 \times 1) = 1{,}000 \times 0.94 = \mathbf{940}$

b. $PV_0 = 1{,}000(1 - 0.06 \times 10) = 1{,}000 \times 0.4 = \mathbf{400}$

c. $PV_0 = 1{,}000(1 - 0.06 \times 20) = 1{,}000 \times (-0.20) = -200$

But the inverse of d is less than 20:

$$\frac{1}{0.06} = 16.67$$

Therefore, we cannot use the simple discount rate of 6% to find the present value of a payment to be made in 20 years, and **there is no solution**.

2.02

Present Value and Accumulated Value Under Simple Discount

The present value is the accumulated value times a discount factor:

$$PV_0 = AV_t(1 - dt)$$

where:

AV_t = Accumulated value at time t

PV_0 = Present value at time 0

d = Simple discount rate per unit of time

We can rearrange the formula in the Key Concept above to find the accumulated value of a present value:

$$PV_0 = AV_t(1 - dt)$$

$$AV_t = \frac{PV_0}{(1 - dt)}$$

Example 2.06 The simple discount rate is 4% per year. How much money would Adam need to deposit now to have $12,870 in 8 years?

Solution The amount to be deposited now is the present value of $10,000:

$$PV_0 = AV_t(1 - dt) = 12{,}870(1 - 0.04 \times 8) = 12{,}870 \times 0.68 = \mathbf{8{,}751.60}$$

The next example verifies the result from the preceding example.

Example 2.07 The simple discount rate is 4% per year. If Adam deposits $8,751.60 now, what is his accumulated value at the end of 8 years?

Solution The accumulated value is:

$$AV_8 = \frac{PV_0}{(1 - 8d)} = \frac{8{,}751.60}{1 - 8 \times 0.04} = \frac{8{,}751.60}{0.68} = \mathbf{12{,}870}$$

If the time interval is not an integer, then discount is credited for the partial unit of time.

Example 2.08 The simple discount rate is 4% per year. Adam will receive $8,000 in 30 months. What is the present value of the payment?

Solution Since the simple discount rate is expressed as an annual value, we use years as our unit of time. 30 months is equal to 2.5 years, and the present value is:

$$PV_0 = AV_t(1 - dt) = 8{,}000(1 - 0.04 \times 2.5) = \mathbf{7{,}200}$$

2.03 Equivalent Simple Interest and Discount Rates

For a given time interval of length t, we can find equivalent simple interest and discount rates. Let's set the accumulated value under simple interest equal to the accumulated value under simple discount:

$$PV_0(1 + it) = \frac{PV_0}{(1 - dt)}$$

$$1 + it = \frac{1}{(1 - dt)}$$

2.03

Equivalent Simple Interest and Simple Discount Rates

If a loan is to be repaid with a single payment at time t, then the equivalent simple interest and discount rates satisfy:

$$1 + it = \frac{1}{(1 - dt)}$$

Example 2.09 Kate borrows $1,000 now and will repay the loan with a payment of $2,000 in 20 years.

a. What is the simple interest rate?

b. What is the simple discount rate?

Solution The simple interest rate is greater than the simple discount rate because the interest rate applies to the beginning balance (i.e., the present value), while the discount rate applies to the larger ending balance (i.e., the accumulated value):

a. The simple interest rate is found below:

$$AV_t = PV_0(1 + it)$$
$$2{,}000 = 1{,}000(1 + 20i)$$
$$i = \mathbf{0.05}$$

b. The simple discount rate is found below:

$$PV_0 = AV_t(1 - dt)$$
$$1{,}000 = 2{,}000(1 - 20d)$$
$$d = 0.025$$

Alternatively, we can convert the simple interest rate into the equivalent simple discount rate for a 20-year loan using the Key Concept above:

$$1 + it = \frac{1}{(1 - dt)}$$
$$1 + 0.05 \times 20 = \frac{1}{1 - 20d}$$
$$d = \mathbf{0.025}$$

The example below shows that to convert a simple interest rate into a simple discount rate, we must know the term of the loan.

Example 2.10 Alan borrows \$1,000 now and will repay the loan with a payment of \$1,500 in 10 years.

a. What is the simple interest rate?

b. What is the simple discount rate?

Solution Although the simple interest rate is 5%, as in the example above, the discount rate is not the same as in the example above because the term of the loan is not the same:

a. The simple interest rate is found below:

$$AV_t = PV_0(1 + it)$$
$$1{,}500 = 1{,}000(1 + 10i)$$
$$i = \mathbf{0.05}$$

b. The simple discount rate is found below:

$$PV_0 = AV_t(1 - dt)$$
$$1{,}000 = 1{,}500(1 - 10d)$$
$$d = 0.033$$

Alternatively, we can convert the simple interest rate into the equivalent simple discount rate for a 10-year loan using the Key Concept above:

$$1 + it = \frac{1}{(1 - dt)}$$
$$1 + 0.05 \times 10 = \frac{1}{1 - 10d}$$
$$d = \mathbf{0.033}$$

In Section 3.04, we will see that under compound interest, we do not need to know the term of the loan to find a discount rate that is equivalent to the interest rate.

2.04 Equations of Value under Simple Interest and Simple Discount

To determine the value of a payment (also known as a **cash flow**), we need to know when it occurs and when it is being valued. Using the timing of a cash flow and the applicable interest or discount rate to find its value is known as taking the time value of money into account.

What are cash flows? A cash flow is a payment of cash. Payments are positive cash flows for the recipient and negative cash flows for the payer. For example, if Larry lends \$1,000 to Mike today, and Mike repays Larry with \$800 in one year and \$400 in two years, then the cash flows to Larry and Mike are as follows:

Time	*Larry's Cash Flows*	*Mike's Cash Flows*
0	*–1,000*	*1,000*
1	*800*	*–800*
2	*400*	*–400*

Example 2.11 On January 31, 2015, \$100 is deposited into a bank account earning simple interest at a rate of 8% per year. What is the value of the deposit on January 31, 2016?

Solution The value of the deposit after one year is found using an equation of value as of January 31, 2016. We can treat January 31, 2015 as time 0, and since January 31, 2016 is one year late, it is time 1:

$$AV_1 = PV_0 \times (1 + it) = 100 \times (1 + 0.08 \times 1) = \mathbf{108}$$

Don't be confused by the conversion of the calendar date into a numeric time. We are just assigning the earlier date to time 0 and counting the number of units of time (in this case, years) between the two calendar dates.

In the example above, the depositor can have \$100 now or \$108 in one year. At a simple interest rate of 8% per year, the two choices are equivalent.

An equation of value allows us to equate one set of cash flows with another at a particular point in time. In the example above, we used the simple interest rate to equate the January 31, 2015 deposit with the January 31, 2016 accumulated value. The equation of value equated the values on January 31, 2016.

Example 2.12 Evan will receive \$$X$ in 2 years. Celeste will receive \$400 in 1 year. At a simple rate of interest of 6% per year, the present values of their payments are equal. Determine X.

Solution An equation of value at time 0 equates the present value of Evan's payment with the present value of Celeste's payment:

$$PV_0 = \frac{400}{1 + 0.06 \times 1}$$
$$PV_0 = 377.36$$

The present value of Evan's payment is \$377.36. We accumulate this value for two years in order to obtain X:

$$X = AV_2 = PV_0 \times (1 + it) = 377.36 \times (1 + 0.06 \times 2) = \mathbf{422.64}$$

The given information above tells us that the present values are equal, so we know that an equation of value holds at time 0. Under simple interest, however, this does not imply that an equation of value holds for other times. For example, if Celeste lends her \$400 for one year at a simple interest rate of 6% per year, then at time 2 she has:

$$400 \times 1.06 = 424$$

This is clearly not equal to the payment of $422.64 that Evan receives at time 2. Therefore, even though we have an equation of value equating the present value of Celeste's payment and Evan's payment at time 0, the equation of value doesn't hold at time 2.

This is in contrast to compound interest, which is discussed in the next chapter. Under compound interest, if an equation of value can be used to equate two sets of payments at one point in time, then an equation of value can be used to equate the two sets of payments at any point in time.

2.05 Questions

Question 2.01

The simple interest rate is 7% per year. If James deposits \$10,000 at the end of 18 months, what is the accumulated value at the end of 5 years?

A 2,450 B 3,500 C 10,000 D 12,450 E 13,500

Question 2.02

The simple interest rate is 7% per year. Kevin makes a deposit of $\$X$ now, which accumulates to \$10,000 at the end of 8 years. Calculate X.

A 4,400 B 5,596 C 5,600 D 6,410 E 6,944

Question 2.03

The simple discount rate is 7% per year. Kevin makes a deposit of $\$X$ now, which accumulates to \$10,000 at the end of 8 years. Calculate X.

A 4,400 B 5,596 C 5,600 D 6,410 E 6,944

Question 2.04

Larry receives a medical bill that is due to be paid in full within 45 days.

If Larry pays within 15 days, he receives a 3% discount.

Larry has two choices:

1. Pay the bill in 15 days for 97% of the amount due in 45 days.
2. In 15 days, deposit 97% of the amount due in 45 days. The deposit will earn an annual simple discount rate of d for 30 days, at which time Larry will use the proceeds to pay the bill in full.

There are 365 days in a year.

What is the minimum value of d that will cause Larry to choose the second option?

A 36.00% B 36.50% C 37.11% D 37.63% E 42.58%

Question 2.05

Consider the following five accumulated values:

- A deposit of \$1,000 at an annual simple interest rate of 5% accumulates to A at the end of 5 years.
- A deposit of \$900 at an annual simple discount rate of 5% accumulates to B at the end of 5 years.
- A deposit of \$900 at an annual simple discount rate of 4% accumulates to C at the end of 6 years.
- A deposit of \$1,000 at an annual simple interest rate of 6% accumulates to D at the end of 4 years.
- A deposit of \$1,000 at an annual simple discount rate of 3% accumulates to E at the end of 8 years.

Which of the following accumulated values is highest: A, B, C, D, or E?

A Amount A B Amount B C Amount C D Amount D E Amount E

Question 2.06

Eric deposits \$100 for t years at an annual simple interest rate of 5%.

Judy deposits \$100 for t years at an annual simple discount rate of 4%.

At the end of t years, Eric's accumulated value is equal to X, and Judy's accumulated value is also equal to X. You are given that $t > 0$.

Calculate X.

A 105 B 115 C 125 D 150 E 175

Question 2.07

The amount paid to repay a loan at time t is the same regardless of whether the loan accrues at an annual simple interest rate of i or an annual simple discount rate of d. Which of the following is a valid expression of t?

I. $t = \dfrac{i-d}{id}$

II. $t = \dfrac{i+d}{id}$

III. $t = \dfrac{i-d+id}{id}$

A I only B II only C III only D I and III only E None is valid

Question 2.08

Ann deposits \$800 into a fund earning an annual simple interest rate of 6%.

Mike deposits \$$X$ into a fund earning an annual simple interest rate of 9%.

Each year, Ann earns twice the interest that Mike does. Calculate X.

A 133.33 B 266.67 C 400.00 D 533.33 E 1,066.67

Question 2.09

Tyler and Haley both invest 1,000 for a period of length T at an annual rate of x.

Tyler's account earns an annual simple interest rate of x.

Haley's account earns an annual simple discount rate of x.

You are given:

- $0 < x < 0.10$
- $0 < T < 10$

Which of the following statements about the final account values is true?

A Haley's account value is certain to be greater than Tyler's account value.

B Tyler's account value is certain to be greater than Haley's account value.

C If $x < \dfrac{1}{T}$, then Tyler's account value is certain to be greater than Haley's account value.

D If $x > \dfrac{0.5}{T}$, then Tyler's account value is certain to be greater than Haley's account value.

E None of the above

Question 2.10

A bank loans Jeff \$1,000 at an annual simple discount rate of 5%. The bank requires that the loan must be repaid within 10 years.

He puts the \$1,000 into a fund that earns an annual simple interest rate of 6%.

At time t, he withdraws the accumulated value in the fund and repays the loan, leaving him with \$$X$:

$$X = \begin{pmatrix} \text{Accumulated value} \\ \text{in fund at time } t \end{pmatrix} - \begin{pmatrix} \text{Accumulated value} \\ \text{of loan at time } t \end{pmatrix}$$

Jeff chooses t to maximize X.

Calculate X.

A 1.74 B 4.39 C 9.11 D 104.55 E 4,390.89

Chapter 3: Compound Interest and Discount

3.01 Compound Interest

Under simple interest, the interest rate is applied only to the original principal, and the interest itself does not earn interest. Under compound interest, the interest rate is applied to the original principal plus any interest earned thus far. Another way to say this is that the interest is added to (or credited to) the accumulated value at the end of each unit of time. When interest is earned on prior interest, we say that the interest rate is a compound interest rate.

Consider a loan where the interest earned over each unit of time is added to the balance of the loan, so that interest is thereafter earned on that interest. Since the original principal of a loan is equal to the present value of the loan, let's use PV_0 to denote the original loan amount.

At time 1, the accumulated value is the original principal plus interest:

$$AV_1 = PV_0 + PV_0 \times i = PV_0 \times (1+i)$$

At time 2, the accumulated value is the time 1 value plus interest:

$$AV_2 = PV_0 \times (1+i) + PV_0 \times (1+i) \times i = PV_0 \times (1+i)(1+i) = PV_0 \times (1+i)^2$$

At time 3, the accumulated is the time 2 value plus interest:

$$AV_3 = PV_0 \times (1+i)^2 + PV_0 \times (1+i)^2 \times i = PV_0 \times (1+i)^2(1+i) = PV_0 \times (1+i)^3$$

At time t, the accumulated value is the time $(t-1)$ value plus interest:

$$AV_t = PV_0 \times (1+i)^{t-1} + PV_0 \times (1+i)^{t-1} \times i = PV_0 \times (1+i)^{t-1}(1+i) = PV_0 \times (1+i)^t$$

Example 3.01

Bill deposits $10,000 into a bank account that pays 3% compound interest per year. The interest is added to the balance at the end of each year and the entire balance earns interest thereafter. For the first 3 years determine the interest earned in each year, and the accumulated value at the end of each year.

Solution

The interest earned in the first year is:

$$10{,}000 \times 0.03 = \mathbf{300}$$

The accumulated value at the end of the first year is the original deposit plus the interest:

$$AV_1 = 10{,}000 + 300 = \mathbf{10{,}300}$$

The interest earned in the second year is:

$$10{,}300 \times 0.03 = \mathbf{309}$$

The accumulated value at the end of the second year is the accumulated value at the end of the first year plus the interest earned in the second year:

$$AV_2 = 10{,}300 + 309 = \mathbf{10{,}609}$$

The interest earned in the third year is:

$$10{,}609 \times 0.03 = \mathbf{318.27}$$

The accumulated value at the end of the third year is the accumulated value at the end of the second year plus the interest earned in the third year:

$$AV_3 = 10{,}609 + 318.27 = \mathbf{10{,}972.27}$$

The year 3 account value can also be found using the formula derived just before the example:

$$AV_t = PV_0 \times (1+i)^t = 10,000 \times (1.03)^3 = 10,927.27$$

When an annual interest rate is compounded annually it is known as an **annual effective interest rate**. The interest rate in the example above is therefore an annual effective interest rate.

Henceforward in this text, unless stated otherwise, we will work with compound interest rates.

Under compound interest, the accumulation function is:

$$a(t) = \frac{\text{Accumulated Value}}{\text{Present Value}} = (1+i)^t$$

3.01

Present Value and Accumulated Value Under Compound Interest

The accumulated value is the present value times the accumulation function:

$$AV_t = PV_0(1+i)^t$$

where:

AV_t = Accumulated value at time t

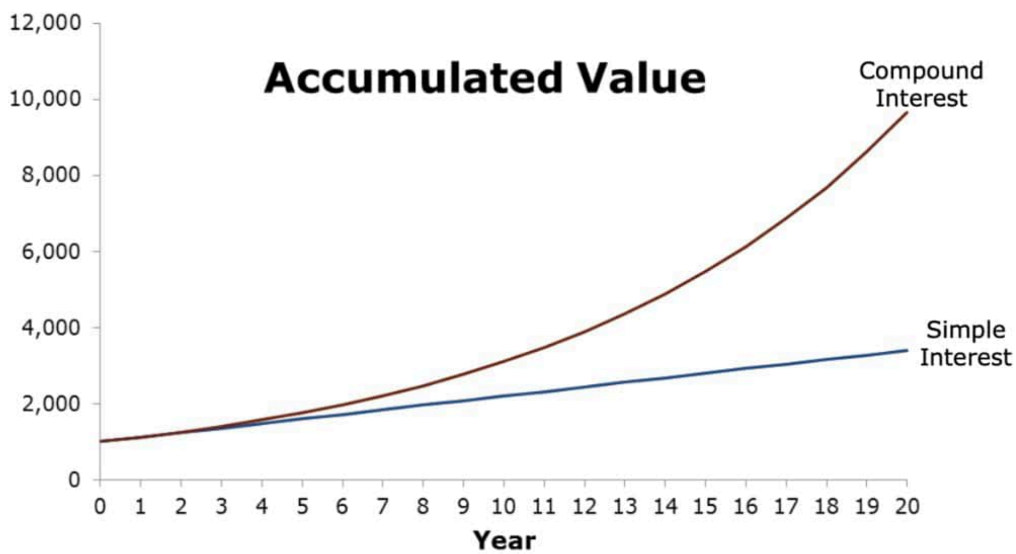
PV_0 = Present value at time 0

i = Compound interest rate per unit of time

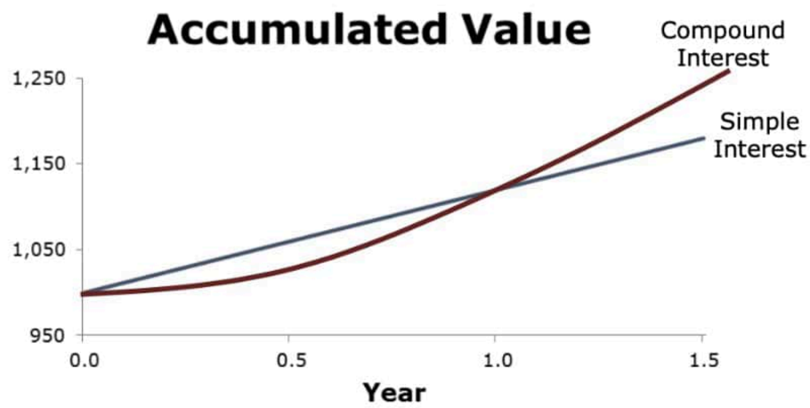
A more general form of the formula above is given by:

$$AV_{t_2} = PV_{t_1}(1+i)^{(t_2-t_1)} \qquad \text{where: } t_1 < t_2$$

After the first year, the accumulated value grows more rapidly under compound interest than under simple interest. The graph below shows how $100 increases under a 12% simple interest rate and a 12% compound interest rate.



Although the graph above makes it appear that the accumulated value is always higher under compound interest, during the first year, it is actually a little less than the accumulated value under simple interest. The graph below, which is not drawn to scale, zooms in on the first year to show that between time 0 and time 1, the accumulated value is greater under simple interest.



The graph shows that the accumulated value under simple interest is greater than or equal to the accumulated value under compound interest during the first year:

$$1 + it \geq (1+i)^t \quad \text{when } t \leq 1$$

$$1 + it < (1+i)^t \quad \text{when } t > 1$$

To find the present value of an accumulated value under compound interest, we rearrange the formula from the Key Concept above:

$$AV_t = PV_0(1+i)^t$$

$$PV_0 = \frac{AV_t}{(1+i)^t}$$

Example 3.02 Sarah will receive \$1,500 in 10 years. The annual effective interest rate is 6%. Find the present value of the payment.

Solution The present value is:

$$PV_0 = \frac{AV_t}{(1+i)^t} = \frac{1{,}500}{1.06^{10}} = \mathbf{837.59}$$

The interest earned over an interval of length t is the difference between the accumulated value and the present value:

$$\text{Interest} = AV_t - PV_0$$

If t is not an integer, then interest is credited for the partial unit of time. That is, the formula above still applies.

Example 3.03 Diane deposits \$400 into an account that earns an annual effective interest rate of 5%. What is the accumulated value in the account at the end of 33 months?

Solution Since the interest rate is an annual interest rate, we convert the 33 months into years:

$$\frac{33}{12} = 2.75$$

The accumulated value at the end of 33 months is:

$$AV_{2.75} = PV_0(1+i)^{2.75} = 400(1.05)^{2.75} = \mathbf{457.44}$$

The one-year accumulation factor is $(1+i)$, and the **one-year discount factor** is its inverse. This discount factor appears frequently, so for convenience we shorten it to v:

$$v = \frac{1}{1+i}$$

We can discount a cash flow for t years by multiplying the cash flow by v^t.

When we use "discount" as a verb, as in the sentence above, we are referring to finding the present value. We can find the present value using interest rates or discount rates.

The present value formula can be written as:

$$PV_0 = \frac{AV_t}{(1+i)^t} = AV_t \times v^t$$

3.02 Current Value Under Compound Interest

The current value of a set of cash flows at time t is the accumulated value of the cash flows occurring prior to time t plus any payment made at time t plus the present value of the cash flows occurring after time t:

$$CV_t = AV_t(\text{payments occurring before } t) + Pmt_t + PV_t(\text{payments occurring after } t)$$

where:

CV_t = Current value at time t of a set of payments

If there are no cash flows occurring before the specified time, then all of the cash flows are discounted back to that point in time, and the current value is a present value. A present value is always calculated by discounting cash flows back to the specified time.

If there are no cash flows occurring after the specified time, then all of the cash flows are accumulated and the current value is an accumulated value (also known as a future value). An accumulated value always accumulates cash flows to a specified time.

Both present values and accumulated values are current values, but the converse is not necessarily the case.

The point in time at which cash flows are valued can be in the past, present, or future, which means that present values are not necessarily valued in the present, current values are not necessarily current, and future values are not necessarily valued in the future. For example, we could discuss the present value on 1/1/1900 of a payment to be on 1/1/1920, but this "present value" is a value in the past. It would be more accurate to describe present values as discounted values, current values as time values, and future values as accumulated values.

This text uses the terms "present value" and "current value" because they are widely used in practice, but we frequently use "accumulated value" in place of the less commonly used "future value."

Given the current value of a set of payments, we can find a subsequent current value by multiplying the original current value by the accumulation function:

$$CV_{t_2} = CV_{t_1}(1+i)^{(t_2-t_1)}$$

The equation above is the general form of the relationship between current values. In the Key Concept below, we assume that the earlier time, t_1, is 0.

3.02

Current Values Under Compound Interest

The current values under compound interest are related as follows:

$$CV_t = CV_0(1+i)^t$$

where:

CV_t = Current value at time t

CV_0 = Current value at time 0

i = Compound interest rate per unit of time

Example 3.04 Fred's grandmother gives him \$100 now and promises him another \$200 to be paid in 4 years. The annual effective interest rate is 5%. Determine the current value of the payments.

a. Now.

b. In 1 year.

c. In 4 years.

Solution a. The current value at the outset (now) is a present value:

$$CV_0 = PV_0 = 100 + 200v^4 = 100 + \frac{200}{1.05^4} = \mathbf{264.54}$$

b. The current value in 1 year is the accumulated value of the \$100 payment plus the present value of the \$200 payment:

$$CV_1 = 100(1+i) + 200v^3 = 100(1.05) + \frac{200}{1.05^3} = \mathbf{277.77}$$

Alternatively, we can accumulate the time 0 current value to time 1:

$$CV_1 = CV_0(1.05)^1 = 264.54 \times 1.05 = \mathbf{277.77}$$

c. The current value in 4 years is the accumulated value of the \$100 payment plus the \$200 payment:

$$CV_4 = 100(1+i)^4 + 200 = 100(1.05)^4 + 200 = \mathbf{321.55}$$

Alternatively, we can accumulate the time 0 current value to time 4, or we can accumulate the time 1 current value to time 4:

$$CV_4 = CV_0(1.05)^4 = 264.54 \times 1.05^4 = \mathbf{321.55}$$

$$CV_4 = CV_1(1.05)^3 = 277.77 \times 1.05^3 = \mathbf{321.55}$$

3.03 Compound Discount

Compound discount applies a discount factor of $(1-d)$ to the accumulated value to find the value one unit of time earlier. This can be done repeatedly to obtain the discounted value at time 0 of the accumulated value at time *t*.

Example 3.05 A payment of \$4,000 will be made in 3 years. The annual compound discount rate is 4%. Determine the value of the payment at the following times:

a. At the end of 2 years.

b. At the end of 1 years.

c. Now.

Solution a. To find the value at time 2 of a payment made at time 3, we discount the time 3 payment for one year:

$$PV_2 = 4,000(1-d) = 4,000(1-0.04) = 4,000 \times 0.96 = \mathbf{3,840}$$

b. To find the time 1 value, we discount the time 2 value for one year:

$$PV_1 = PV_2(1-d) = 3,840(1-0.04) = 3,840 \times 0.96 = \mathbf{3,686.40}$$

c. To find the time 0 value, we discount the time 1 value for one year:

$$PV_0 = PV_1(1-d) = 3,686.40(1-0.04) = 3,686.40 \times 0.96 = \mathbf{3,538.94}$$

In the example above, the value now can also be found by multiplying the payment amount by a 3-year discount factor:

$$PV_0 = 4,000(1-d)^3 = 4,000(0.96)^3 = 3,538.94$$

Since $(1-d)$ is the one-year discount factor, it is equal to v:

$$v = 1 - d$$

3.03

Present Value and Accumulated Value Under Compound Discount

The present value is the accumulated value times a discount factor:

$$PV_0 = AV_t(1-d)^t$$

where:

AV_t = Accumulated value at time t

PV_0 = Present value at time 0

d = Compound discount rate per unit of time

More generally, the equation above can also be written as:

$$PV_{t_1} = AV_{t_2}(1-d)^{(t_2-t_1)} \qquad \text{where: } t_1 < t_2$$

Even more generally, it can be written in terms of current values:

$$CV_{t_1} = CV_{t_2}(1-d)^{(t_2-t_1)}$$

To use the compound discount rate to find the accumulated value of a present value, we rearrange the formula in the above Key Concept:

$$PV_0 = AV_t(1-d)^t$$

$$AV_t = \frac{PV_0}{(1-d)^t}$$

Example 3.06 Diane deposits \$400 into an account that earns a compound discount rate of 5% per year. What is the accumulated value in the account at the end of 4 years?

Solution The accumulated value is:

$$AV_4 = \frac{PV_0}{(1-d)^4} = \frac{400}{(1-0.05)^4} = \mathbf{491.10}$$

If t is not an integer, then interest is credited for a partial unit of time. That is, the formula above still applies.

Example 3.07 Diane deposits \$400 into an account that earns a compound discount rate of 5% per year. What is the accumulated value in the account at the end of 33 months?

Solution Since the discount rate is an annual discount rate, we convert the 33 months into years:

$$\frac{33}{12} = 2.75$$

The accumulated value at the end of 33 months is:

$$AV_{2.75} = \frac{PV_0}{(1-d)^{2.75}} = \frac{400}{(1-0.05)^{2.75}} = \mathbf{460.60}$$

As before, the interest (or discount) earned over an interval of length t is the difference between the accumulated value and the present value:

$$\text{Interest} = \text{Discount} = AV_t - PV_0$$

3.04 Equivalent Compound Interest and Discount Rates

Two rates of interest or discount are **equivalent** if they produce the same present values and accumulated values. We can find equivalent compound interest and compound discount rates. Let's set the accumulated value under compound interest equal to the accumulated value under compound discount:

$$PV_0(1+i)^t = \frac{PV_0}{(1-d)^t}$$

$$(1+i)^t = \frac{1}{(1-d)^t}$$

$$1+i = \frac{1}{1-d}$$

Notice that the relationship between i and d does not depend on t. This means that a compound interest rate can be converted into a unique, equivalent compound discount rate, and this relationship will hold regardless of the length of the time interval considered.

We can express i in terms of d:

$$1+i = \frac{1}{1-d} \quad \Rightarrow \quad i = \frac{1}{1-d} - 1 = \frac{1}{1-d} - \frac{1-d}{1-d} = \frac{d}{1-d}$$

We can also express d in terms of i:

$$1+i = \frac{1}{1-d} \quad \Rightarrow \quad \frac{1}{1+i} = 1-d \quad \Rightarrow \quad d = 1 - \frac{1}{1+i} = \frac{1+i}{1+i} - \frac{1}{1+i} = \frac{i}{1+i}$$

3.04

Converting between *i* and *d*

The compound interest rate and the compound discount rate are related as follows:

$$i = \frac{d}{1-d} \qquad d = \frac{i}{1+i} = iv$$

The first expression below implies that a compound discount rate is less than its equivalent compound interest rate:

$$d = \frac{i}{1+i} \quad \Rightarrow \quad d < i$$

This isn't surprising since the interest (or discount) earned over one unit of time can be calculated by applying i to the present value at the beginning of the unit of time or by applying d to the accumulated value one unit of time later.

For the rates to be equivalent, the interest and the discount must be equal, and since the present value is less than the accumulated value, the interest rate must be greater than the discount rate:

$$\text{Interest} = i \times PV_0 \quad \& \quad \text{Discount} = d \times AV_1$$

$$\text{Interest} = \text{Discount}$$

$$i \times PV_0 = d \times AV_1 \quad \& \quad PV_0 < AV_1 \quad \Rightarrow \quad d < i$$

Example 3.08

The annual effective interest rate is 6%.

a. Find the annual compound discount rate.

b. Find the value of the discount factor, v.

Solution

a. The discount rate is:

$$d = \frac{i}{1+i} = \frac{0.06}{1.06} = \mathbf{0.05660}$$

b. The discount factor can be found in two ways:

$$v = \frac{1}{1+i} = \frac{1}{1.06} = \mathbf{0.94340}$$

$$v = (1-d) = 1 - 0.05660 = \mathbf{0.94340}$$

3.05 Nominal Interest Rates

We've defined i to be the interest rate per unit of time when the interest is credited at the end of each unit of time. Another way to say this is that i is the interest rate when the interest is credited once per unit of time.

Let's define $i^{(m)}$ to be the **nominal interest rate** per unit of time when the interest is credited m times per unit of time. Each unit of time contains m periods of length $1/m$, and the interest rate applicable to each period is called the periodic effective interest rate. This periodic effective interest rate is the nominal interest rate per unit of time divided by m:

$$\frac{i^{(m)}}{m}$$

The period, $1/m$, is called the **interest conversion period,** the **interest period,** or the **conversion period**.

If the unit of time is 1 year, then $i^{(12)}$ is a nominal annual interest rate, and $\frac{i^{(12)}}{12}$ is a monthly effective interest rate:

Unit of time $=$ 1 year

Interest period $=$ 1 month

In practice, the unit of time is usually one year.

Present Value and Accumulated Value Using a Nominal Interest Rate

3.05 The accumulated value is the present value times an accumulation factor:

$$AV_t = PV_0\left(1+\frac{i^{(m)}}{m}\right)^{mt}$$

where:

AV_t = Accumulated value at time t

PV_0 = Present value at time 0

$i^{(m)}$ = Nominal interest rate compounded m times per unit of time

We refer to $i^{(m)}$ as the nominal interest rate, compounded m times per unit of time. If, as is often the case, the unit of time is one year, then we can say that $i^{(m)}$ is the nominal annual interest rate compounded m times per year. The **compounding frequency** is m. A nominal interest rate must be divided by its compounding frequency to convert it into an effective rate that can be used in calculations. Since all compound interest rates are nominal interest rates, the descriptor "nominal" is sometimes left off, and $i^{(m)}$ can also be referred to simply as the interest rate, compounded m times per year.

The words "compounded" and "convertible" have the same meaning when describing interest rates, so we can also say that $i^{(m)}$ is the nominal interest rate, convertible m times per unit of time.

If $m = 1$, then the interest rate is compounded once per unit of time, and we have:

$$i^{(1)} = i$$

When the unit of time is one year, it is common to refer to i as the annual effective interest rate or the effective annual interest rate.

If m is equal to 2, 4, 6 or 12, then we can replace "m times per year" with semiannually, quarterly, bimonthly, or monthly respectively.

Bi means every other, while semi means twice. Therefore, bimonthly means every other month, while semiannually means twice per year.

Unfortunately, bi can also be used to mean twice. Therefore, technically, bimonthly could mean twice per month. This usage is less common, and clarifying language should be included prior to such usage. Another complication is that that biennial means every other year. Since biennial means every other year, some say that biannual always means twice per year, even without clarifying language.

Example 3.09 The nominal annual interest rate compounded quarterly is 10%. If $2,000 is deposited now, what is the accumulated value after 3.5 years?

Solution The interest rate is compounded 4 times per year, and the quarterly effective interest rate is:

$$\frac{0.10}{4} = 0.025$$

The payment is accumulated for 3.5 years, which is 14 quarters:

$$\frac{3.5}{0.25} = 14$$

The accumulated value is:

$$AV_{3.5} = PV_0\left(1+\frac{i^{(m)}}{m}\right)^{mt} = 2{,}000\left(1+\frac{0.10}{4}\right)^{4\times 3.5} = 2{,}000(1.025)^{14} = \mathbf{2{,}825.95}$$

When the number of interest conversion periods is not an integer, the Key Concept above still applies.

Example 3.10 The nominal annual interest rate compounded quarterly is 10%. If $2,000 is deposited now, what is the accumulated value after 3 years and 2 months?

Solution The payment is accumulated for 3 years and 2 months, which is 12.6667 quarters:

$$\frac{3+\frac{2}{12}}{0.25} = \frac{3.1667}{0.25} = 12.6667$$

The accumulated value is:

$$AV_{3.1667} = 2{,}000\left(1+\frac{0.10}{4}\right)^{4\times 3.1667} = 2{,}000(1.025)^{12.6667} = \mathbf{2{,}734.42}$$

To use a nominal interest rate to find the present value of an accumulated value, we rearrange the formula in the Key Concept above:

$$PV_0 = \frac{AV_t}{\left(1+\frac{i^{(m)}}{m}\right)^{mt}}$$

Example 3.11 The nominal annual interest rate convertible monthly is 12%. Find the present value of a payment of $10,000 to be made at the end of 5 years.

Solution The interest conversion period is one month, and the monthly effective interest rate is:

$$\frac{0.12}{12} = 0.01$$

The payment is discounted for 5 years, which is 60 months:

$$PV_0 = \frac{AV_t}{\left(1+\frac{i^{(m)}}{m}\right)^{mt}} = \frac{AV_5}{\left(1+\frac{i^{(12)}}{12}\right)^{12\times 5}} = \frac{10{,}000}{\left(1+\frac{0.12}{12}\right)^{60}} = \frac{10{,}000}{(1.01)^{60}} = \mathbf{5{,}504.50}$$

The value of m doesn't necessarily have to be an integer.

Example 3.12 The nominal annual interest rate convertible biennially is 12%. Find the present value of a payment of \$10,000 to be made at the end of 5 years.

Solution The interest conversion period is 2 years, and m is equal to 0.5. The 2-year interest rate is:

$$\frac{0.12}{0.5} = 0.24$$

The payment is discounted for 5 years, which is 2.5 interest conversion periods

$$\frac{5}{2} = 2.5$$

The present value is:

$$PV_0 = \frac{AV_t}{\left(1+\frac{i^{(m)}}{m}\right)^{mt}} = \frac{10{,}000}{\left(1+\frac{0.12}{0.5}\right)^{0.5\times 5}} = \frac{10{,}000}{(1.24)^{2.5}} = \mathbf{5{,}840.44}$$

We can use one interest rate compounded at one frequency to find an equivalent interest rate compounded at another frequency. Consider two equivalent interest rates, one compounded m times per unit of time and the other compounded p times per unit of time. Since they are equivalent, they both produce the same accumulated value, if they begin with the same present value:

$$AV_t = PV_0\left(1+\frac{i^{(m)}}{m}\right)^{mt} \quad \& \quad AV_t = PV_0\left(1+\frac{i^{(p)}}{p}\right)^{pt}$$

$$PV_0\left(1+\frac{i^{(m)}}{m}\right)^{mt} = PV_0\left(1+\frac{i^{(p)}}{p}\right)^{pt}$$

$$\left(1+\frac{i^{(m)}}{m}\right)^{mt} = \left(1+\frac{i^{(p)}}{p}\right)^{pt}$$

$$\left(1+\frac{i^{(m)}}{m}\right)^{m} = \left(1+\frac{i^{(p)}}{p}\right)^{p}$$

3.06

Equivalent Nominal Interest Rates

If $i^{(m)}$ is equivalent to $i^{(p)}$, then the accumulation factor for one unit of time is:

$$\left(1+\frac{i^{(m)}}{m}\right)^{m} = \left(1+\frac{i^{(p)}}{p}\right)^{p}$$

If we set $p = 1$, then the relationship above becomes:

$$1+i = \left(1+\frac{i^{(m)}}{m}\right)^{m}$$

Solving for the nominal interest rate, we have:

$$i^{(m)} = \left[(1+i)^{1/m} - 1\right] \times m$$

When the interest rates are annual rates, the nominal interest rate is known as the **annual percentage rate** (APR) and the annual effective interest rate is known as the **annual percentage yield** (APY).

$$i^{(m)} = \text{APR} = \text{Annual Percentage Rate} = \text{Nominal annual interest rate}$$
$$i = \text{APY} = \text{Annual Percentage Yield} = \text{Annual effective interest rate}$$

Banks and other financial institutions frequently use the terms APR and APY to describe the interest rates offered on their products.

Example 3.13 The nominal annual interest rate compounded 4 times per year is 8%. Calculate the equivalent nominal interest rate that is compounded:

a. once per year
b. twice per year
c. monthly

Solution

a. The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(4)}}{4}\right)^4 - 1 = \left(1 + \frac{0.08}{4}\right)^4 - 1 = (1.02)^4 - 1 = \mathbf{0.0824}$$

b. Let's use the annual effective interest rate to find the equivalent annual interest rate that is compounded semiannually:

$$i^{(2)} = \left[(1+i)^{1/m} - 1\right] \times m = \left[1.0824^{0.5} - 1\right] \times 2 = \mathbf{0.0808}$$

c. The equivalent interest rate that is compounded monthly is:

$$i^{(12)} = \left[(1+i)^{1/m} - 1\right] \times m = \left[1.0824^{1/12} - 1\right] \times 12 = \mathbf{0.0795}$$

In the example above, we found the annual effective interest rate first, and then converted it into rates that are compounded semiannually and monthly.

In the next example, we obtain the same results, but in each case we start with the quarterly accumulation factor of 1.02.

Example 3.14 The nominal annual interest rate compounded 4 times per year is 8%. Calculate the equivalent nominal interest rate that is compounded:

a. once per year
b. twice per year
c. monthly

Solution Since the quarterly effective interest rate is 2%, the quarterly accumulation factor is 1.02.

a. There are 4 quarters in one year, so the annual accumulation factor is the quarterly accumulation factor taken to the power of 4:

$$1 + i = 1.02^4$$
$$i = \mathbf{0.0824}$$

b. There are 2 quarters in a six-month period, so the semi-annual accumulation factor is the square of the quarterly accumulation factor:

$$1 + \frac{i^{(2)}}{2} = 1.02^2$$
$$i^{(2)} = \mathbf{0.0808}$$

c. There are 3 months within a quarter, so the monthly accumulation factor is the cube root of the quarterly accumulation factor:

$$1+\frac{i^{(12)}}{12}=1.02^{1/3}$$
$$i^{(12)}=\mathbf{0.0795}$$

The example above provides an intuitive feel for how equivalent interest rates relate to one another. The main idea is to identify an accumulation factor for a time interval and then determine the exponent to use with that accumulation factor to obtain an accumulation factor for a different time interval.

3.06 Nominal Discount Rates

We've defined d to be the discount rate per unit of time when the discount is deducted once per unit of time.

Let's define $d^{(m)}$ to be the **nominal discount rate** per unit of time when the discount is deducted m times per unit of time. Each unit of time contains m conversion periods of length $1/m$, and the discount rate applicable to each period is called the periodic effective discount rate. This periodic effective discount rate is the nominal discount rate per unit of time divided by m:

$$\frac{d^{(m)}}{m}$$

If the unit of time is 1 year, then $d^{(12)}$ is the nominal annual discount rate, and $\frac{d^{(12)}}{12}$ is the monthly effective discount rate.

Present Value and Accumulated Value Using a Nominal Discount Rate

3.07 The present value is the accumulated value times a discount factor:

$$PV_0=AV_t\left(1-\frac{d^{(m)}}{m}\right)^{mt}$$

where:

AV_t = Accumulated value at time t

PV_0 = Present value at time 0

$d^{(m)}$ = Nominal discount rate compounded m times per unit of time

We refer to $d^{(m)}$ as the nominal discount rate compounded m times per unit of time, where m is the compounding frequency. If, as is often the case, the unit of time is one year, then we can say that $d^{(m)}$ is the annual discount rate compounded m times per year. A nominal discount rate must be divided by its compounding frequency to convert it into an effective rate that can be used in calculations. Since all compound discount rates are nominal discount rates, the descriptor "nominal" is sometimes left off, and $d^{(m)}$ can also be referred to simply as the discount rate compounded m times per year.

The words "compounded" and "convertible" have the same meaning when describing discount rates, so we can also say that $d^{(m)}$ is the nominal discount rate convertible m times per unit of time.

Example 3.15 The nominal annual discount rate convertible monthly is 6%. Find the present value of a payment of $10,000 to be made at the end of 4 years.

Solution The effective monthly discount rate is:

$$\frac{0.06}{12} = 0.005$$

The present value is:

$$PV_0 = AV_t\left(1 - \frac{d^{(m)}}{m}\right)^{mt} = 10{,}000\left(1 - \frac{d^{(12)}}{12}\right)^{12\times 4} = 10{,}000 \times (0.995)^{48} = \mathbf{7{,}861.54}$$

To use a nominal discount rate to find the accumulated value of a present value, we rearrange the formula above:

$$AV_t = \frac{PV_0}{\left(1 - \frac{d^{(m)}}{m}\right)^{mt}}$$

Example 3.16 The nominal annual discount rate convertible quarterly is 8%. A loan of $3,000 is to be paid off with a single payment at the end of 10 months. Determine the amount of that payment.

Solution The accumulated value of the loan is:

$$AV_t = \frac{PV_0}{\left(1 - \frac{d^{(m)}}{m}\right)^{mt}} = \frac{3{,}000}{\left(1 - \frac{0.08}{4}\right)^{4\times\frac{10}{12}}} = \frac{3{,}000}{(0.98)^{3.33}} = \mathbf{3{,}208.98}$$

If $m = 1$, then the discount rate is compounded once per unit of time, and it is equal to the **annual effective discount rate** d:

$$d^{(1)} = d$$

If we change the compounding frequency of the discount rate, then we must change the discount rate as well in order for the new discount rate to be equivalent to the original discount rate. We can use a discount rate compounded at one frequency to find an equivalent discount rate compounded at another frequency.

Consider two equivalent discount rates, one compounded m times per unit of time and the other compounded p times per unit of time. Since they are equivalent, they both produce the same present value if they begin with the same accumulated value:

$$PV_0 = AV_t\left(1 - \frac{d^{(m)}}{m}\right)^{mt} \quad \& \quad PV_0 = AV_t\left(1 - \frac{d^{(p)}}{p}\right)^{pt}$$

$$AV_t\left(1 - \frac{d^{(m)}}{m}\right)^{mt} = AV_t\left(1 - \frac{d^{(p)}}{p}\right)^{pt}$$

$$\left(1 - \frac{d^{(m)}}{m}\right)^{mt} = \left(1 - \frac{d^{(p)}}{p}\right)^{pt}$$

$$\left(1 - \frac{d^{(m)}}{m}\right)^{m} = \left(1 - \frac{d^{(p)}}{p}\right)^{p}$$

Equivalent Nominal Discount Rates

3.08 If $d^{(m)}$ is equivalent to $d^{(p)}$, then the discount factor for one unit of time is:

$$\left(1-\frac{d^{(m)}}{m}\right)^m = \left(1-\frac{d^{(p)}}{p}\right)^p$$

If we set $p=1$, then we have an expression for v:

$$v = 1-d = \left(1-\frac{d^{(m)}}{m}\right)^m$$

Solving for the nominal annual discount rate, we have:

$$d^{(m)} = \left[1-(1-d)^{1/m}\right]\times m = \left[1-v^{1/m}\right]\times m$$

Example 3.17 The nominal annual discount rate compounded 4 times per year is 8%. Calculate the equivalent nominal discount rate that is compounded:

a. once per year
b. twice per year
c. monthly

Solution a. The annual effective discount rate is:

$$d = 1-\left(1-\frac{d^{(m)}}{m}\right)^m = 1-\left(1-\frac{0.08}{4}\right)^4 = 1-(0.98)^4 = \mathbf{0.0776}$$

b. Let's use the annual effective discount rate to find the equivalent annual discount rate that is compounded semiannually:

$$d^{(2)} = \left[1-(1-d)^{1/m}\right]\times m = \left[1-(1-0.0776)^{0.5}\right]\times 2 = \mathbf{0.0792}$$

c. The equivalent discount rate that is compounded monthly is:

$$d^{(12)} = \left[1-(1-d)^{1/m}\right]\times m = \left[1-(1-0.0776)^{1/12}\right]\times 12 = \mathbf{0.0805}$$

In the example above, we found the annual effective discount rate first, and then converted it into rates that are compounded semiannually and monthly.

In the next example, we obtain the same results, but in each case we start with the quarterly discount factor of 0.98.

Example 3.18 The nominal annual discount rate compounded 4 times per year is 8%. Calculate the equivalent nominal discount rate that is compounded:

a. once per year
b. twice per year
c. monthly

Solution Since the quarterly effective discount rate is 2%, the quarterly discount factor is 0.98:

$$v^{0.25} = \left(1-\frac{d^{(4)}}{4}\right) = \left(1-\frac{0.08}{2}\right) = 0.98$$

a. There are 4 quarters in one year, so the annual discount factor is the quarterly discount factor taken to the power of 4:

$$1-d = 0.98^4$$
$$d = \mathbf{0.0776}$$

b. There are 2 quarters in a six-month period, so the semi-annual discount factor is the square of the quarterly discount factor:

$$1-\frac{d^{(2)}}{2}=0.98^2$$
$$d^{(2)}=\mathbf{0.0792}$$

c. There are 3 months within a quarter, so the monthly discount factor is the cube root of the quarterly discount factor:

$$1-\frac{d^{(12)}}{12}=0.98^{1/3}$$
$$d^{(12)}=\mathbf{0.0805}$$

The example above provides an intuitive feel for how equivalent discount rates relate to one another. The main idea is to identify a discount factor for a time interval and then determine the exponent to use with that discount factor to obtain a discount factor for a different time interval.

3.07 Equivalent Nominal Interest and Discount Rates

We can find equivalent nominal interest and nominal discount rates. Let's set the accumulated value under compound interest equal to the accumulated value under compound discount:

$$PV_0\left(1+\frac{i^{(m)}}{m}\right)^{mt}=\frac{PV_0}{\left(1-\frac{d^{(p)}}{p}\right)^{pt}}$$

$$\left(1+\frac{i^{(m)}}{m}\right)^{mt}=\left(1-\frac{d^{(p)}}{p}\right)^{-pt}$$

$$\left(1+\frac{i^{(m)}}{m}\right)^{m}=\left(1-\frac{d^{(p)}}{p}\right)^{-p}$$

As with i and d, the relationship between $i^{(m)}$ and $d^{(p)}$ does not depend on t.

3.09

Converting Between Different Nominal Rates

The nominal interest rates and the nominal discount rates are related as follows:

$$1+i=\left(1+\frac{i^{(m)}}{m}\right)^{m}=\left(1-\frac{d^{(p)}}{p}\right)^{-p}=(1-d)^{-1}$$

Example 3.19

The nominal interest rate compounded 2 times per year is 9%. Calculate the equivalent nominal discount rate compounded quarterly.

Solution

We obtain the answer in two ways.

First, we use the formula in the Key Concept above:

$$\left(1+\frac{i^{(2)}}{2}\right)^{2}=\left(1-\frac{d^{(4)}}{4}\right)^{-4}$$

$$\left(1+\frac{0.09}{2}\right)^{2}=\left(1-\frac{d^{(4)}}{4}\right)^{-4}$$

$$d^{(4)}=\mathbf{0.087072}$$

Second, we begin by noting that the accumulation factor for a 6-month interval is 1.045:

$$\left(1+\frac{i^{(2)}}{2}\right)=\left(1+\frac{0.09}{2}\right)=1.045$$

The discount factor for a 6-month interval is therefore the inverse of 1.045. There are two quarters in a 6-month period, so the square of the 3-month discount factor is equal to the 6-month discount factor:

$$\left(1-\frac{d^{(4)}}{4}\right)^2=\frac{1}{1.045}$$

$$d(4)=\mathbf{0.087072}$$

Let's verify that the two rates found above are equivalent.

Example 3.20

Calculate the accumulated value of $1,000 that is lent for 3 years:

a. if the nominal interest rate compounded semiannually is 9%

b. if the nominal discount rate compounded quarterly is 8.7072%.

Solution

a. The accumulated value using the nominal interest rate of 9% is:

$$AV_3=PV_0\left(1+\frac{i^{(2)}}{2}\right)^{2\times 3}=1{,}000\times\left(1+\frac{0.09}{2}\right)^6=\mathbf{1{,}302.26}$$

b. The accumulated value using the nominal discount rate of 8.7072% is:

$$AV_3=\frac{PV_0}{\left(1-\frac{d^{(m)}}{m}\right)^{mt}}=\frac{1{,}000}{\left(1-\frac{0.087072}{4}\right)^{4\times 3}}=\mathbf{1{,}302.26}$$

If a nominal interest rate is convertible at the same frequency as a nominal discount rate, then we can simplify the expression above to relate the periodic effective interest rate directly to the periodic effective discount rate:

$$1+\frac{i^{(m)}}{m}=\left(1-\frac{d^{(m)}}{m}\right)^{-1}$$

We can solve for the periodic effective interest rate:

$$\frac{i^{(m)}}{m}=\left(1-\frac{d^{(m)}}{m}\right)^{-1}-1=\frac{1}{1-\frac{d^{(m)}}{m}}-\frac{1-\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}}=\frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}}$$

We can also solve for the periodic effective discount rate:

$$\frac{d^{(m)}}{m}=1-\left(1+\frac{i^{(m)}}{m}\right)^{-1}=\frac{1+\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}-\frac{1}{1+\frac{i^{(m)}}{m}}=\frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}$$

To summarize, the effective periodic interest rates have the following relationships:

$$\frac{i^{(m)}}{m}=\frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}} \qquad \frac{d^{(m)}}{m}=\frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}$$

The relationships between the periodic effective rates above are analogous to the relationships that we saw earlier for annual effective rates:

$$i = \frac{d}{1-d} \qquad\qquad d = \frac{i}{1+i}$$

Example 3.21 The nominal interest rate compounded quarterly is 9%. Calculate the equivalent nominal discount rate compounded quarterly.

Solution We use the quarterly effective interest rate to find the quarterly effective discount rate:

$$\frac{d^{(4)}}{4} = \frac{\frac{i^{(4)}}{4}}{1+\frac{i^{(4)}}{4}} = \frac{\frac{0.09}{4}}{1+\frac{0.09}{4}} = 0.0220$$

The nominal discount rate is the found by annualizing the quarterly effective discount rate:

$$d^{(4)} = 4 \times 0.0220 = \mathbf{0.0880}$$

3.08 Equations of Value Revisited

In Chapter 2, we observed that under simple interest, equations of value can hold for one point in time and not hold for another. Under compound interest and compound discount, if an equation of value for two sets of cash flows holds at one point in time, then it holds for all points in time.

Example 3.22 Evan will receive $\$X$ in 2 years. Celeste will receive \$400 in 1 year. At an annual effective interest rate of 6% per year, the present values of their payments are equal.

a. Determine X.

b. Find the accumulated value of Celeste's payment at time 2.

Solution a. An equation of value at time 0 equates the present value of Evan's payment with the present value of Celeste's payment:

$$PV_0 = \frac{400}{(1+0.06)^1}$$

$$PV_0 = 377.36$$

The present value of Evan's payment is \$377.36. We accumulate this value for two years in order to obtain X:

$$X = AV_2 = PV_0 \times (1+i)^t = 377.36 \times (1+0.06)^2 = \mathbf{424}$$

b. The accumulated value of Celeste's payment at time 2 is:

$$400 \times (1+0.06) = \mathbf{424}$$

Since the value of Celeste's payment and the value of Evan's payment are equal when valued at time 0, the value of their payments is equal when valued at any point in time. Part b of the above example shows that the value of the payments is equal when valued at time 2.

3.09 Describing Compound Interest and Compound Discount Rates

Suppose that we are told that funds will be accumulated at 12%. That alone is not enough information to allow us to determine the rate of growth of the funds. Before we can use an interest or discount rate, we must know:

Whether the rate is an interest rate or a discount rate.

The unit of time in which the rate is expressed.

The length of the conversion period.

In Section 16.04, we'll see that we will also need to know whether the rate is a spot rate, forward rate, or yield, but for now we are assuming that those rates are equal to each other.

Example 3.23 The nominal annual discount rate compounded quarterly is 12%. Calculate the present value of $1,300 to be paid in 17 months.

Solution Let's begin by answering 3 questions about the rate of 12%:

1. Is the rate an interest rate or a discount rate? Discount rate.
2. What is the unit of time in which the rate is expressed? One year.
3. What is the conversion period? 3 months.

With these questions answered, we can now find the present value:

$$PV_0 = AV_t\left(1-\frac{d^{(m)}}{m}\right)^{mt} = 1{,}300\left(1-\frac{0.12}{4}\right)^{4\times 17/12} = \mathbf{1{,}093.91}$$

If we use one year as our unit of time, then the interest and discount rates below can be described as shown to the right. The descriptions below are not exhaustive, but they provide some idea of how interest and discount rates are frequently described.

$i^{(1)} = i$	Annual effective interest rate Effective annual interest rate Nominal annual interest rate convertible 1 time per year Annual percentage yield (APY)
$i^{(m)}$	Nominal interest rate convertible *m* times per year Nominal interest rate compounded *m* times per year Annual interest rate convertible *m* times per year Interest rate convertible *m* times per year Annual percentage rate (APR)
$i^{(2)}$	Nominal interest rate convertible 2 times per year Annual interest rate convertible twice per year Annual interest rate compounded semiannually
$i^{(4)}$	Nominal interest rate convertible 4 times per year Annual interest rate compounded quarterly
$i^{(12)}$	Nominal interest rate convertible 12 times per year Annual interest rate compounded monthly
$\frac{i^{(m)}}{m}$	Periodic effective interest rate for a period of length 1/*m*
$\frac{i^{(2)}}{2}$	Periodic effective interest rate for a 6-month period Semiannual effective interest rate
$\frac{i^{(4)}}{4}$	Periodic effective interest rate for a 3-month period Quarterly effective interest rate
$\frac{i^{(12)}}{12}$	Periodic effective interest rate for a period of one month Monthly effective interest rate

$d^{(1)} = d$	Annual effective discount rate Nominal annual discount rate convertible 1 time per year
$d^{(m)}$	Nominal discount rate convertible m times per year Nominal discount rate compounded m times per year Annual discount rate convertible m times per year Discount rate convertible m times per year
$d^{(2)}$	Nominal discount rate convertible 2 times per year Annual discount rate convertible twice per year Annual discount rate compounded semiannually
$d^{(4)}$	Nominal discount rate convertible 4 times per year Annual discount rate compounded quarterly
$d^{(12)}$	Nominal discount rate convertible 12 times per year Annual discount rate compounded monthly
$\frac{d^{(m)}}{m}$	Periodic effective discount rate for a period of length $1/m$
$\frac{d^{(2)}}{2}$	Periodic effective discount rate for a 6-month period Semiannual effective discount rate
$\frac{d^{(4)}}{4}$	Periodic effective discount rate for a 3-month period Quarterly effective discount rate
$\frac{d^{(12)}}{12}$	Periodic effective discount rate for a period of one month Monthly effective discount rate

3.10 Questions

Question 3.01

The parents of 4 children, ages 1, 4, 6, and 9 wish to set up a trust fund that will pay A to each child upon attainment of age 17 and B to each child upon attainment of age 22.

The trust fund will be funded by a single payment. Which of the following is the correct value of the single payment?

A $\dfrac{A}{v^{16}+v^{13}+v^{11}+v^{8}}+\dfrac{B}{v^{21}+v^{18}+v^{16}+v^{13}}$

B $A\left[v^{16}+v^{13}+v^{11}+v^{8}\right]-B\left[v^{21}+v^{18}+v^{16}+v^{13}\right]$

C $A\left[v^{21}+v^{18}+v^{16}+v^{13}\right]+B\left[v^{16}+v^{13}+v^{11}+v^{8}\right]$

D $\left[A+Bv^{5}\right]\left[v^{16}+v^{13}+v^{11}+v^{8}\right]$

E $\left[Av^{5}+B\right]\left[v^{16}+v^{13}+v^{11}+v^{8}\right]$

Question 3.02

A retailer is having a sale and allows its customers to choose between the following two options:

(i) The customer can pay 92% of the purchase price in 9 months, or

(ii) The customer can pay now and take X% off the purchase price.

A customer is indifferent between the two choices when they are valued using an annual effective interest rate of 7%.

Calculate X.

A 3.2 B 4.9 C 8.7 D 12.6 E 14.0

Question 3.03

Gail can receive one of the following two payment streams:

(i) 100 at time 0, 200 at time n years, and 300 at time $2n$ years

(ii) 604.42 at time $(n + 2)$ years

At an annual effective interest rate of i, the present values of the two payment streams are equal.

You are given that $v^n = 0.7$ and $v = \dfrac{1}{1+i}$.

Calculate i, the annual effective rate of interest.

A 3.31% B 4.36% C 4.56% D 8.53% E 9.33%

Question 3.04

Gail can receive one of the following two payment streams:

(i) 100 at time 0, 200 at time n years, and 300 at time $2n$ years

(ii) 604.42 at time $(n + 2)$ years

At an annual effective interest rate of i, the present values of the two payment streams are equal.

You are given that $v^n = 0.7$ and $v = \frac{1}{1+i}$, where i is the annual effective rate of interest.

Calculate n.

A 4 B 5 C 6 D 7 E 8

Question 3.05

Wanda and Claire each open new bank accounts at time 0.

Wanda deposits 1,000 into her bank account, and Claire deposits \$700 into hers. Each account earns the same annual effective interest rate, i.

The amount of interest earned in Wanda's account during the 11th year is equal to X. The amount of interest earned in Claire's account during the 15th year is also equal to X.

Calculate X.

A 159.25 B 227.50 C 243.92 D 248.71 E 266.67

Question 3.06

Bonnie deposits \$1,800 into a savings account at time 0. The savings account pays simple interest at an annual rate of i.

Clyde deposits \$1,000 into a different savings account at time 0. Clyde's savings account pays interest at an annual effective rate of i.

Bonnie and Clyde earn the same amount of interest during the 7th year.

Calculate i.

A 6.6% B 7.6% C 8.8% D 9.9% E 10.3%

Question 3.07

The present value of a payment of 1,553.50 at the end of T months is equal to the present value of the following payments:

At the end of	Amount
1 month	300
18 months	500
24 months	700

The effective annual interest rate is 4%.

Calculate T.

A 28 B 29 C 30 D 31 E 32

Question 3.08

Lucy has two savings accounts. Savings account #1 earns interest at 3% effective annually and savings account #2 earns interest at 5% effective annually. Lucy has not made any deposits or withdrawals since July 1, 2007, when the amount in savings account #1 was three times the amount in savings account #2.

The sum of the two accounts on July 1, 2016 was 150,000.

Determine the sum of the two accounts on July 1, 2007.

A 27,444	B 82,332	C 109,777	D 113,684	E 137,221

Question 3.09

At an annual effective interest rate i, $i > 0$, the following are equal:

(i) The present value of 12,000 at the end of 12 years

(ii) The sum of the present values of 3,000 at the end of year t and 63,000 at the end of year $2t$

(iii) 8,000 immediately

Calculate the present value of a payment of 6,000 at the end of year $(t + 1)$.

A 1,934	B 2,000	C 2,210	D 2,900	E 3,867

Question 3.10

Jill deposits 40 today and another 20 in six years into a fund paying simple interest of 5% per year.

Tom will make the same two deposits, but the 40 will be deposited n years from today and the 20 will be deposited $2n$ years from today. Tom's deposits earn an annual effective rate of 4%.

At the end of 15 years, the accumulated amount of Jill's deposits equals the accumulated amount of Tom's deposits.

Calculate n.

A 1.4	B 1.7	C 2.0	D 2.4	E 2.5

Question 3.11

Sarah will receive $10,000 in 3 years and $15,000 in 5 years. The compound discount rate is 6.5% per year.

Calculate the present value of today of the payments.

A 10,719	B 12,357	C 18,893	D 19,227	E 19,461

Question 3.12

Linda deposits $8,000 into an account now and $2,000 in 2 years. Interest is credited at an annual discount rate of 7%.

Calculate the balance in the account at the end of 6 years.

A 9,730	B 10,312	C 14,627	D 14,677	E 15,039

Question 3.13

Bill's parents have deposited \$2,000 into an account that will earn an annual effective interest rate of 8% for 4 years and 6 months, at which time Bill will be given the accumulated value in the fund.

Bill uses an annual effective discount rate of 5% per year to find the present value of the payment he will receive in 4 years and 6 months.

Find the present value calculated by Bill.

A 1,762 B 2,245 C 2,270 D 3,562 E 3,625

Question 3.14

A deposit of X is made into a fund that pays an annual effective interest rate of 3% for 12 years.

At the same time, $X/3$ is deposited into another fund that pays an annual effective rate of discount of d for 12 years.

The amounts of interest earned over the 12 years are equal for both funds.

Calculate d.

A 2.0% B 6.6% C 7.1% D 7.5% E 8.7%

Question 3.15

At an annual effective discount rate of d, $d > 0$, each of the following two sets of payments has a present value that is equal to K:

(i) A payment of 169 immediately and another payment of 169 at the end of 1 year.

(ii) A payment of 196 at the end of 2 years and another payment of 196 at the end of 3 years.

Calculate K.

A 315 B 326 C 351 D 378 E 472

Question 3.16

Which of the expressions below is FALSE?

A $i(1+i) = \dfrac{d}{v - vd}$

B $i^2 = \dfrac{d^2}{v^2}$

C $id = i - d$

D $i - d = \dfrac{1 - v - iv^2}{v}$

E $i + d = i(1 - v)$

Question 3.17

The annual interest rate convertible monthly is 12%. Calculate the equivalent annual effective interest rate.

A 11.39% B 12.00% C 12.12% D 12.68% E 12.75%

Question 3.18

The annual interest rate convertible monthly is 12%. Calculate the equivalent two-year effective interest rate.

A 24.00% B 25.37% C 25.44% D 25.50% E 26.97%

Question 3.19

Patty deposits $1,800 into a savings account at time 0. The savings account pays simple interest at an annual rate of i.

Sally deposits $1,000 into a different savings account at time 0. Sally's savings account pays interest at an annual nominal rate of i compounded quarterly.

Patty and Sally earn the same amount of interest during the last 3 months of the 7^{th} year.

Calculate i.

A 8.80% B 8.88% C 9.10% D 11.16% E 36.39%

Question 3.20

Calculate the nominal annual rate of interest convertible monthly that is equivalent to a nominal rate of interest of 12% per year convertible quarterly.

A 9.90% B 11.82% C 11.88% D 12.55% E 15.79%

Question 3.21

Wanda and Claire each open up new bank accounts at time 0.

Wanda deposits 1,000 into her bank account, and Claire deposits $700 into hers. Each account earns the same nominal annual interest rate compounded monthly.

The amount of interest earned in Wanda's account during the 11^{th} year is equal to X. The amount of interest earned in Claire's account during the 15^{th} year is also equal to X.

Calculate X.

A 18.19 B 18.96 C 218.31 D 227.50 E 761.69

Question 3.22

Sam deposits D into a savings account at time 0, and the account pays interest at a nominal rate of i, compounded semiannually.

Dennis deposits $2D$ into a different savings account at time 0, and this account pays interest at a simple annual interest rate of i.

Sam and Dennis each earn the same amount of interest during the last 6 months of the 7^{th} year.

Calculate i.

A 0.00% B 5.56% C 9.53% D 10.95% E 11.25%

Question 3.23

Heidi and Adam each take out a loan of L.

Heidi will repay her loan by making a payment of 1,400 in 20 years.

Adam will repay his loan by making a payment of 2,000 in 20 years.

The nominal semiannual interest rate charged to Heidi is half the nominal semiannual interest rate charged to Adam.

Calculate L.

A 483 B 683 C 688 D 690 E 977

Question 3.24

Michelle deposits $100 into an account that earns interest at an annual rate of 12% convertible quarterly.

At the same time, Lucy deposits $100 into an account that earns interest at an annual rate of 6% convertible monthly.

For both accounts, interest is deposited into the account only at the end of the account's interest conversion period.

Calculate the number of months until the amount in Michelle's account is at least twice as much as the amount in Lucy's account.

A 141 B 142 C 143 D 144 E 147

Question 3.25

Larry plans to deposit $1,500 n months from now and $3,000 $2n$ months from now into a fund.

The fund will earn interest at a nominal annual rate of 9% compounded monthly.

Larry needs to have at least $5,900 in the fund in six years.

Calculate the maximum integral value of n such that Larry will have at least $5,900 six years from today.

A 20 B 21 C 22 D 23 E 24

Question 3.26

Vicky's parents have deposited $2,000 into an account that will earn a nominal annual interest rate of 8% compounded semiannually for 4 years and 6 months, at which time Vicky will be given the accumulated value in the fund.

Vicky uses a nominal annual discount rate of 5% convertible monthly to find the present value of the payment she will receive in 4 years and 6 months.

Find the present value calculated by Vicky.

A 1,811 B 2,172 C 2,179 D 2,272 E 4,506

Question 3.27

Deb plans to deposit $1,500 n months from now and $3,100 $2n$ months from now into a fund.

Interest is credited at a nominal discount rate of 9% compounded monthly.

Deb needs the fund to have at least $5,900 in the fund in six years.

Calculate the maximum integral value of n such that Deb will have at least $5,900 six years from today.

A 20 B 21 C 22 D 23 E 24

Question 3.28

Calculate the nominal annual rate of interest convertible monthly that is equivalent to a nominal rate of discount of 22% per year convertible monthly.

A 19.6% B 20.1% C 22.4% D 24.4% E 24.9%

Question 3.29

Calculate the nominal annual rate of discount convertible quarterly that is equivalent to a nominal rate of rate interest of 14% per year convertible monthly.

A 12.9% B 13.0% C 13.7% D 13.8% E 18.1%

Question 3.30

Let A be the accumulated value of 50 invested for four years at a nominal annual rate of discount of d convertible quarterly, which is equivalent to an annual effective interest rate of i.

Let B be the accumulated value of 50 invested for 8 years at a nominal annual rate of discount d convertible semiannually.

You are given that:

$$\frac{A}{B} = \left(\frac{51}{52}\right)^{16}$$

Calculate i.

A 7.55% B 7.92% C 8.00% D 8.16% E 8.42%

Chapter 4: Constant Force of Interest

4.01 Deriving the Force of Interest

The **force of interest** is defined as the instantaneous change in the accumulated value per unit of the accumulated value. We call the force of interest r:

$$r = \frac{\frac{d(AV_t)}{dt}}{AV_t}$$

Some textbooks prefer to call the force of interest δ, but it is also common for textbooks to reserve δ to denote the dividend yield of common stock. To avoid confusion when encountering textbooks that use δ as the dividend yield, we use r to denote the force of interest.

If you take the actuarial Exam FM, then you might encounter a question that refers to the force of interest as δ on the exam.

Suppose that the accumulated value earns interest at a nominal rate of $i^{(m)}$. If there is an initial deposit at time 0 of X, then the accumulated value at time t is:

$$AV_t = X\left(1+\frac{i^{(m)}}{m}\right)^{mt}$$

Let's determine the force of interest when the interest rate is $i^{(m)}$:

$$r = \frac{\frac{d(AV_t)}{dt}}{AV_t} = \frac{\frac{d\left(X\left(1+\frac{i^{(m)}}{m}\right)^{mt}\right)}{dt}}{X\left(1+\frac{i^{(m)}}{m}\right)^{mt}} = \frac{\frac{d\left(\left[\left(1+\frac{i^{(m)}}{m}\right)^{m}\right]^{t}\right)}{dt}}{\left(1+\frac{i^{(m)}}{m}\right)^{mt}}$$

$$= \frac{\left(1+\frac{i^{(m)}}{m}\right)^{mt}\ln\left[\left(1+\frac{i^{(m)}}{m}\right)^{m}\right]}{\left(1+\frac{i^{(m)}}{m}\right)^{mt}} = \ln\left[\left(1+\frac{i^{(m)}}{m}\right)^{m}\right]$$

As shown above, the force of interest is equal to the natural log of the one period accumulation factor. We can use both sides of the equation below as exponent for e:

$$r = \ln\left[\left(1+\frac{i^{(m)}}{m}\right)^{m}\right] \qquad \Rightarrow \qquad e^{r} = \left(1+\frac{i^{(m)}}{m}\right)^{m}$$

4.02 The Force of Interest and Equivalent Rates

Based on the relationships established in Section 3.07, we can now write equations relating the force of interest, interest rates, and discount rates.

4.01

Converting Between Different Nominal Rates

The nominal interest rates and the nominal discount rates are related as follows:

$$e^r = 1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m = \left(1 - \frac{d^{(p)}}{p}\right)^{-p} = (1-d)^{-1}$$

The Key Concept above implies that we can use the force of interest to find an accumulated value:

$$AV_t = PV_0(1+i)^t = PV_0 \times e^{rt}$$

Example 4.01 The force of interest is equal to 8%. If a deposit of \$20 is made now, what is the accumulated value at the end of 5 years?

Solution The accumulated value is:

$$AV_5 = PV_0 \times e^{5r} = 20 \times e^{5 \times 0.08} = \mathbf{29.8365}$$

Example 4.02 The force of interest is equal to 8%. Calculate:

a. the annual effective interest rate.

b. the nominal interest rate convertible quarterly.

c. the nominal discount rate convertible monthly.

d. the annual effective discount rate.

Solution

a. The annual effective interest rate is found below:

$$e^r = 1 + i$$
$$e^{0.08} = 1 + i$$
$$i = \mathbf{0.0833}$$

b. The nominal interest rate convertible quarterly is found below:

$$e^r = \left(1 + \frac{i^{(m)}}{m}\right)^m$$
$$e^{0.08} = \left(1 + \frac{i^{(4)}}{4}\right)^4$$
$$i^{(4)} = \mathbf{0.0808}$$

c. The nominal discount rate convertible monthly is found below:

$$e^r = \left(1 - \frac{d^{(p)}}{p}\right)^{-p}$$
$$e^{0.08} = \left(1 - \frac{d^{(12)}}{12}\right)^{-12}$$
$$d^{(12)} = \mathbf{0.0797}$$

d. The annual effective discount rate is found below:

$$e^r = (1-d)^{-1}$$
$$e^{0.08} = (1-d)^{-1}$$
$$d = \mathbf{0.0769}$$

For a given force of interest, the equivalent nominal interest rate falls as m increases, and the equivalent nominal discount rate increases as p increases. This implies:

$$d < d^{(p)} < r < i^{(m)} < i \text{ for } m > 1 \text{ \& } p > 1$$

For a given nominal interest rate, more frequent compounding increases the accumulated value.

4.03 Force of Interest as a Continuously Compounded Rate

For a fixed nominal interest rate, the force of interest increases as the compounding frequency increases.

Example 4.03 $100 is lent for 5 years and repaid in a single payment at the end of the 5 years. Calculate the amount of the payment if the interest rate is:

a. an annual effective interest rate of 7%.

b. compounded twice per year and is $i^{(2)} = 7\%$.

c. convertible monthly and is $i^{(12)} = 7\%$.

d. the force of interest and is $r = 7\%$.

Solution The accumulated values are shown below:

a. $AV_5 = 100(1.07)^5 = \mathbf{140.26}$

b. $AV_5 = 100\left(1 + \dfrac{0.07}{2}\right)^{2\times 5} = \mathbf{141.06}$

c. $AV_5 = 100\left(1 + \dfrac{0.07}{12}\right)^{12\times 5} = \mathbf{141.76}$

d. $AV_5 = 100 \times e^{0.07\times 5} = \mathbf{141.91}$

Example 4.04 The nominal annual interest rate is 10%. Calculate the equivalent force of interest when the compounding frequency is:

a. once per year

b. quarterly

c. monthly

d. 24 times per year

e. 100 times per year

Solution The force of interest increases with the compounding frequency of the nominal interest rate.

a. $r = \ln(1.10) = \mathbf{0.09531}$

b. $r = \ln\left[\left(1 + \frac{0.10}{4}\right)^4\right] = \mathbf{0.09877}$

c. $r = \ln\left[\left(1 + \frac{0.10}{12}\right)^{12}\right] = \mathbf{0.09959}$

d. $r = \ln\left[\left(1 + \frac{0.10}{24}\right)^{24}\right] = \mathbf{0.09979}$

e. $r = \ln\left[\left(1 + \frac{0.10}{100}\right)^{100}\right] = \mathbf{0.09995}$

The example above shows that for a fixed nominal rate, the force of interest increases as the compounding frequency increases. Let's consider the limit as m approaches infinity:

$$r = \ln\left[\left(1+\frac{i^{(m)}}{m}\right)^m\right]$$

$$r = \underset{m\to\infty}{Lim} \ln\left[\left(1+\frac{i^{(m)}}{m}\right)^m\right] = \underset{m\to\infty}{Lim}\, m\ln\left(1+\frac{i^{(m)}}{m}\right) = \underset{m\to\infty}{Lim}\frac{\ln\left(1+\frac{i^{(m)}}{m}\right)}{\frac{1}{m}}$$

We can apply L'Hospital's rule:

$$r = \underset{m\to\infty}{Lim}\frac{\frac{1}{1+\frac{i^{(m)}}{m}}\times i^{(m)}\times\left(-\frac{1}{m^2}\right)}{-\frac{1}{m^2}} = \underset{m\to\infty}{Lim}\frac{1}{1+\frac{i^{(m)}}{m}}\times i^{(m)} = \underset{m\to\infty}{Lim}\frac{1}{1+\frac{i^{(m)}}{m}}\times i^{(m)}$$

$$= \underset{m\to\infty}{Lim}\, i^{(m)}$$

As the compounding frequency increases to infinity, we say that the interest rate is **continuously compounded**, and the nominal interest rate becomes the force of interest:

$$r = \underset{m\to\infty}{Lim}\, i^{(m)}$$

This also holds for the discount rate:

$$r = \ln\left[\left(1+\frac{i^{(m)}}{m}\right)^m\right] = \ln\left[\left(1-\frac{d^{(m)}}{m}\right)^{-m}\right]$$

$$r = \underset{m\to\infty}{Lim} \ln\left[\left(1-\frac{d^{(m)}}{m}\right)^{-m}\right] = \underset{m\to\infty}{Lim}(-m)\ln\left(1-\frac{d^{(m)}}{m}\right) = \underset{m\to\infty}{Lim}\frac{\ln\left(1-\frac{d^{(m)}}{m}\right)}{-\frac{1}{m}}$$

$$= \underset{m\to\infty}{Lim}\frac{\frac{1}{1-\frac{d^{(m)}}{m}}\times -d^{(m)}\times\frac{-1}{m^2}}{\frac{1}{m^2}} = \underset{m\to\infty}{Lim}\frac{1}{1-\frac{d^{(m)}}{m}}\times d^{(m)} = \underset{m\to\infty}{Lim}\, d^{(m)}$$

An interest rate or a discount rate that is continuously compounded is equal to the force of interest:

$$r = \underset{m\to\infty}{Lim}\, i^{(m)} = \underset{m\to\infty}{Lim}\, d^{(m)}$$

For this reason, the force of interest is also known as the continuously compounded rate of interest or the continuously compounded rate of discount.

4.04 Questions

Question 4.01

The annual effective rate of interest is 8%. Calculate the force of interest.

A 7.41% B 7.70% C 8.00% D 8.33% E 8.70%

Question 4.02

The nominal annual interest rate compounded monthly is 10%. Calculate the force of interest.

A 9.96% 10.04% C 10.47% D 10.52% E 10.60%

Question 4.03

The nominal annual discount rate compounded quarterly is 9%. Calculate the force of interest.

A 8.61% B 8.90% C 9.10% D 9.42% E 9.77%

Question 4.04

The monthly effective discount rate is 1%. Calculate the force of interest.

A 12.00% B 12.06% C 12.12% D 12.68% E 12.81%

Question 4.05

The nominal annual interest rate compounded every other year is 11%. Calculate the annual force of interest.

A 9.94% B 10.44% C 10.71% D 11.30% E 11.63%

Question 4.06

The continuously compounded interest rate is 6%. Calculate the following equivalent rates:

a. the annual effective interest rate
b. the monthly effective interest rate
c. the annual interest rate compounded monthly
d. the annual effective discount rate
e. the quarterly effective discount rate
f. the annual discount rate convertible quarterly

Question 4.07

Susan will receive a payment of $3,000 in 2 years, $8,000 in 5 years, and $10,000 in 7 years.

The annual force of interest is 7%.

Calculate the present value of the payments.

A 14,177 B 14,372 C 14,452 C 14,722 E 14,764

Question 4.08

Charlotte deposits \$2,000 into a fund. Six years later, she deposits another \$8,000.

Four years after her initial deposit, Charlotte withdraws \$1,500 from the fund.

The fund earns a constant force of interest of 7%.

Calculate the balance in the fund ten years after her initial deposit.

A 12,022 B 12,122 C 12,170 D 12,330 E 12,508

Question 4.09

The nominal annual interest rate compounded quarterly is 7%, and the equivalent force of interest is r.

If the force of interest falls to half of its previous value, then what is the new equivalent nominal annual interest rate compounded quarterly?

A 3.40% B 3.42% C 3.44% D 3.46% E 3.48%

Question 4.10

Dolores can receive one of the following two payment streams:

(i) 100 at time 0, 200 at time n years, and 300 at time $2n$ years

(ii) 604.42 at time $(n + 2)$ years

At an annual effective interest rate of i, the present values of the two payment streams are equal.

You are given that $v^n = 0.7$ and $v = \frac{1}{1+i}$.

Calculate r, the continuously compounded interest rate.

A 4.46% B 4.48% C 4.56% D 8.92% E 8.96%

Question 4.11

David can receive one of the following two payment streams:

(i) 100 at time 0, 200 at time n years, and 300 at time $2n$ years

(ii) 600 at time n years

The present values of the two payment streams are equal.

You are given that the annual force of interest is 12.21%.

Calculate n.

A 8.0 B 8.5 C 9.0 D 9.5 E 10.0

Question 4.12

You are given that the following interest rates are equivalent:

i is the annual effective interest rate

$i^{(m)}$ is the nominal annual interest rate compounded m times per year

r is the annual force of interest

d is the annual effective discount rate

$d^{(p)}$ is the nominal annual discount rate compounded p times per year

Which of the following expressions is FALSE?

A $r < i$

B $r > d$

C $d^{(2)} > d$

D $d^{(3)} > d^{(1/2)}$

E $i^{(1/2)} < i$

Question 4.13

Rose deposits \$200 into an account that earns an annual interest rate of 7.5% compounded quarterly.

At the same time, Bill deposits \$220 into an account that earns a constant force of interest of δ.

After 12 years and 1 month, the value in each account is the same.

Calculate δ.

A 6.42% B 6.44% C 6.51% D 6.62% E 6.64%

Question 4.14

Suzie deposits \$200 into an account that earns an annual simple interest rate of 5%.

At the same time, John deposits \$220 into an account that earns a constant force of interest of δ.

After 5 years, the value in each account is the same.

Calculate δ.

A 2.56% B 2.59% C 2.65% D 2.73% E 2.97%

Question 4.15

The derivative of the force of interest with respect to the annual effective interest rate is denoted by:

$$\frac{d}{di}\delta$$

The derivative of the annual effective interest rate with respect to the annual effective discount rate is denoted by:

$$\frac{d}{dd}i$$

Simplify the following expression:

$$\left[\frac{d}{di}\delta\right] \times \left[\frac{d}{dd}i\right]$$

A i B v C d D $1+i$ E v^3

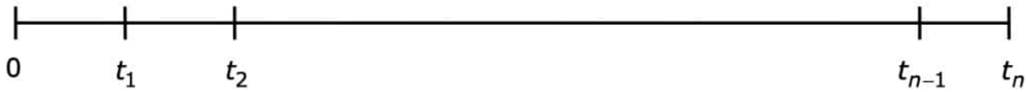
Chapter 5: Varying Rates

5.01 Varying Compound Interest Rates

Interest rates and discount rates can vary across time. Let's use the following notation for an interest rate that is constant from time s to time t:

$$i_{s,t} = \text{Effective interest rate per unit of time that applies from time } s \text{ to time } t$$

Consider a time interval from 0 to t_n that is broken down into n subintervals as shown below:



5.01

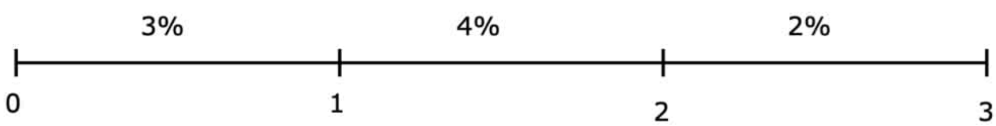
Accumulated Value with Varying Compound Interest Rates

The accumulated value is the present value times a product of accumulation factors:

$$AV_{t_n} = PV_0\left(1+i_{0,t_1}\right)^{t_1}\left(1+i_{t_1,t_2}\right)^{t_2-t_1}\cdots\left(1+i_{t_{n-1},t_n}\right)^{t_n-t_{n-1}}$$

Example 5.01 Over 3 years, the annual effective interest rate is 3%, 4%, and 2%, in the 1st, 2nd, and 3rd years respectively. Calculate the accumulated value at the end of 3 years of $350 that is deposited at the beginning of the first year.

Solution The effective interest rates are shown in the timeline below:



The accumulated value is:

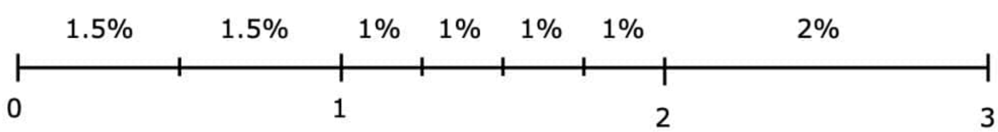
$$AV_3 = 350(1.03)^1(1.04)^{2-1}(1.02)^{3-2} = \mathbf{382.42}$$

The compounding frequency of the interest rates can change over time.

Example 5.02 Over 3 years, the nominal interest rate is 3%, 4%, and 2%, in the 1st, 2nd, and 3rd years respectively. In the first year, the 3% rate is compounded semiannually. In the second year, the 4% rate is compounded quarterly. In the third year, the 2% rate is an annual effective rate.

Calculate the accumulated value at the end of 3 years of $350 that is deposited at the beginning of the first year.

Solution The effective interest rates are shown on the timeline below:



Let's find the annual accumulation factors:

$$\text{First year: } 1+i_{0,1} = \left(1+\frac{0.03}{2}\right)^2$$

$$\text{Second year: } 1+i_{1,2} = \left(1+\frac{0.04}{4}\right)^4$$

$$\text{Third year: } 1+i_{2,3} = 1.02$$

The accumulated value is:

$$AV_{t_n} = PV_0\left(1+i_{0,t_1}\right)^{t_1}\left(1+i_{t_1,t_2}\right)^{t_2-t_1}\cdots\left(1+i_{t_{n-1},t_n}\right)^{t_n-t_{n-1}}$$

$$AV_3 = 350\left[\left(1+\frac{0.03}{2}\right)^2\right]^1\left[\left(1+\frac{0.04}{4}\right)^4\right]^{2-1}\left[(1.02)\right]^{3-2}$$

$$AV_3 = 350\left(1+\frac{0.03}{2}\right)^2\left(1+\frac{0.04}{4}\right)^4(1.02)$$

$$AV_3 = \mathbf{382.72}$$

The notation makes the solution in the example above look more difficult than it really is. Take another look at the second-to-last equation above, which is repeated below:

$$AV_3 = 350\left(1+\frac{0.03}{2}\right)^2\left(1+\frac{0.04}{4}\right)^4(1.02)$$

We can obtain that equation fairly easily by observing the following:

In the first year, there are two 6-month conversion periods, each of which has an interest rate of 0.03/2. Since there are two conversion periods, the first exponent is 2.

In the second year, there are four 3-month conversion periods, each of which has an interest rate of 0.04/4. Since there are four conversion periods, the second exponent is 4.

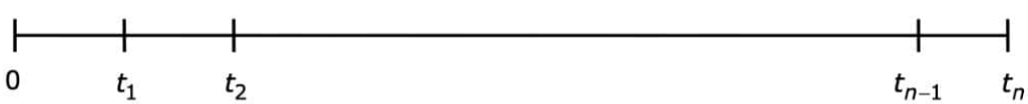
In the third year, there is just one 1-year conversion period, which has an interest rate of 0.02. Since there is one conversion period, the final, implied, exponent is 1.

5.02 Varying Compound Discount Rates

Let's use the following notation for a discount rate that is constant from time s to time t:

$d_{s,t}$ = Effective discount rate per unit of time that applies from time s to time t

For the Key Concept below, we use the same time line as before:



5.02

Present Value with Varying Compound Discount Rates

The present value is the accumulated value times a product of discount factors:

$$PV_0 = AV_{t_n}\left(1-d_{0,t_1}\right)^{t_1}\left(1-d_{t_1,t_2}\right)^{t_2-t_1}\cdots\left(1-d_{t_{n-1},t_n}\right)^{t_n-t_{n-1}}$$

Example 5.03

In year 1, the discount rate convertible semiannually is 6%. In year 2, the discount rate convertible quarterly is 8%. For the first 6 months of year 3, the discount rate convertible monthly is 12%. For the last 6 months of year 3, the annual effective discount rate is 7%.

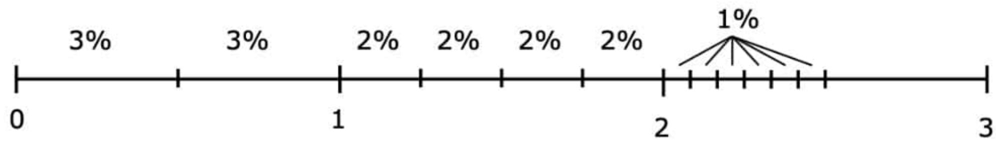
Calculate the accumulated value at the end of 3 years of \$350 deposited at the beginning of the first year.

Solution We can use the equation in the Key Concept above to find the accumulated value instead of the present value:

$$PV_0 = AV_{t_n}\left(1-d_{0,t_1}\right)^{t_1}\left(1-d_{t_1,t_2}\right)^{t_2-t_1}\cdots\left(1-d_{t_{n-1},t_n}\right)^{t_n-t_{n-1}}$$

$$AV_{t_n} = PV_0\left(1-d_{0,t_1}\right)^{-t_1}\left(1-d_{t_1,t_2}\right)^{-(t_2-t_1)}\cdots\left(1-d_{t_{n-1},t_n}\right)^{-(t_n-t_{n-1})}$$

The effective discount rates for the first 2.5 years are shown in the timeline below:



Let's find the annual discount factors that apply over the 3 years in terms of the annual effective discount rates:

First year: $1-d_{0,1} = \left(1-\dfrac{0.06}{2}\right)^2$

Second year: $1-d_{1,2} = \left(1-\dfrac{0.08}{4}\right)^4$

First half of third year: $1-d_{2,2.5} = \left(1-\dfrac{0.12}{12}\right)^{12}$

Second half of third year: $1-d_{2.5,3} = (1-0.07)$

We have:

$$AV_{t_n} = PV_0\left(1-d_{0,t_1}\right)^{-t_1}\left(1-d_{t_1,t_2}\right)^{-(t_2-t_1)}\cdots\left(1-d_{t_{n-1},t_n}\right)^{-(t_n-t_{n-1})}$$

$$AV_3 = 350\left[\left(1-\frac{0.06}{2}\right)^2\right]^{-1}\left[\left(1-\frac{0.08}{4}\right)^4\right]^{-(2-1)}\left[\left(1-\frac{0.12}{12}\right)^{12}\right]^{-(2.5-2)}(1-0.07)^{-(3-2.5)}$$

$$AV_3 = 350(1-0.03)^{-2}(1-0.02)^{-4}(1-0.01)^{-6}(1-0.07)^{-0.5}$$

$$AV_3 = \mathbf{444.19}$$

The notation makes the solution in the example above look more difficult than it really is. Take another look at the second-to-last equation above, which is repeated below:

$$AV_3 = 350(1-0.03)^{-2}(1-0.02)^{-4}(1-0.01)^{-6}(1-0.07)^{-0.5}$$

We can obtain that equation fairly easily by observing the following:

In the first year, there are two 6-month conversion periods, each of which has a discount rate of 0.06/2, which is 0.03. Since there are two conversion periods, the first exponent is -2.

In the second year, there are four 3-month conversion periods, each of which has a discount rate of 0.08/4, which is 0.02. Since there are four conversion periods, the second exponent is -4.

In the first half of the third year, there are six 1-month conversion periods, each of which has a discount rate of 0.12/12, which is 0.01. Since there are six conversion periods, the third exponent is -6.

In the second half of the third year, there are 0.5 1-year conversion periods, which has a discount rate of 0.07. Since there are 0.5 conversion periods, the fourth exponent is -0.5.

5.03 Varying Force of Interest

Let's use the following notation for a continuously compounded interest rate that is constant from time s to time t:

$$r_{s,t} = \text{Force of interest per unit of time that applies from time } s \text{ to time } t$$

We again consider a time interval from 0 to t_n that is broken down into n subintervals as shown below:

5.03

Accumulated Value with Discretely Varying Force of Interest

The accumulated value is the present value times an accumulation factor:

$$AV_{t_n} = PV_0 e^{r_{0,t_1} \times t_1} e^{r_{t_1,t_2} \times (t_2 - t_1)} \cdots e^{r_{t_{n-1},t_n} \times (t_n - t_{n-1})}$$

Example 5.04 Over 3 years, the force of interest is 3%, 4%, and 2%, in the 1st, 2nd, and 3rd years respectively. Calculate the accumulated value at the end of 3 years of $500 that is deposited at the beginning of the first year.

Solution The continuously compounded interest rates are shown in the timeline below:

The accumulated value is:

$$AV_3 = 500e^{0.03 \times 1} e^{0.04 \times (2-1)} e^{0.02 \times (3-2)} = 500e^{0.03} e^{0.04} e^{0.02}$$
$$= 500e^{0.03+0.04+0.02} = 500e^{0.09} = \mathbf{547.09}$$

Consider a continuously compounded interest rate that varies continuously over time:

$$r_t = \text{Force of interest at time } t$$

The force of interest at time t is the instantaneous change in the accumulated value per unit of the accumulated value:

$$r_t = \frac{\frac{d(AV_t)}{dt}}{AV_t}$$

We can also write this as:

$$r_t = \frac{\frac{d(CV_t)}{dt}}{CV_t}$$

An accumulated value at time t is the accumulated value of cash flows occurring on or before time t. A current value is a broader concept, and a current value can be used to value cash flows occurring before, on, or after time t.

The derivative of the natural log of a function is the inverse of the function times the derivate of the function, so we can write the force of interest as shown at left in the expression below:

$$\frac{d\left[\ln(CV_t)\right]}{dt} = \frac{1}{CV_t}\frac{d(CV_t)}{dt} = \frac{\frac{d(CV_t)}{dt}}{CV_t} = r_t$$

As shown above, the force of interest is the derivative of the natural log of the current value:

$$r_t = \frac{d\left[\ln(CV_t)\right]}{dt}$$

Let's integrate both sides from t_1 to t_2:

$$\int_{t_1}^{t_2} r_s ds = \int_{t_1}^{t_2} \frac{d\left[\ln(CV_s)\right]}{ds} ds$$

$$\int_{t_1}^{t_2} r_s ds = \ln(CV_{t_2}) - \ln(CV_{t_1})$$

$$\int_{t_1}^{t_2} r_s ds = \ln\left(\frac{CV_{t_2}}{CV_{t_1}}\right)$$

$$e^{\int_{t_1}^{t_2} r_s ds} = \frac{CV_{t_2}}{CV_{t_1}}$$

The current value at time t_2 is the current value at time t_1 times an accumulation factor:

$$CV_{t_2} = CV_{t_1} \times e^{\int_{t_1}^{t_2} r_s ds}$$

If there are no cash flows before time t_1 or after time t_2, then the earlier current value can be written as a present value and the later one can be written as an accumulated value:

$$AV_{t_2} = PV_{t_1} \times e^{\int_{t_1}^{t_2} r_s ds}$$

5.04

Accumulated Value with Continuously Varying Force of Interest

The accumulated value is the present value times an accumulation factor:

$$AV_{t_2} = PV_{t_1} e^{\int_{t_1}^{t_2} r_s ds} \qquad \text{where } t_1 < t_2$$

In the example below, the continuously compounded interest rate is a linear function of t.

Example 5.05

The force of interest depends on the time at which it is evaluated, measured in years:

$$0.08 - 0.02t \qquad \text{where } 0 < t \le 4$$

Calculate the accumulated value at the end of 4 years of $1,000 that is deposited at the beginning of the first year.

Solution

Let's begin by integrating the force of interest from time 0 to time 4:

$$\int_{t_1}^{t_2} r_s ds = \int_0^4 (0.08 - 0.02s) ds = 0.08s - 0.01s^2\Big|_0^4 = 0.08(4) - 0.01(4)^2 = 0.16$$

The accumulated value is:

$$AV_4 = 1,000 \times e^{\int_0^4 (0.08-0.02s)ds} = 1,000 \times e^{0.16} = \mathbf{1,173.51}$$

To find the present value of a future value, we can rearrange the formula in the Key Concept:

$$PV_{t_1} = AV_{t_2} e^{-\int_{t_1}^{t_2} r_s ds}$$

Example 5.06 The force of interest depends on the time at which it is evaluated, measured in years:

$$0.08 - 0.02t \qquad \text{where } 0 < t \leq 4$$

Calculate the present value of a payment of \$1,173.51 that is made at the end of 4 years.

Solution Let's begin by integrating the force of interest from time 0 to time 4:

$$\int_{t_1}^{t_2} r_s ds = \int_0^4 (0.08 - 0.02s)ds = 0.08s - 0.01s^2 \Big|_0^4 = 0.08(4) - 0.01(4)^2 = 0.16$$

The present value is:

$$PV_0 = 1,173.51 \times e^{-\int_0^4 (0.08-0.02s)ds} = 1,173.51 \times e^{-0.16} = \mathbf{1,000.00}$$

5.04 Mix of Varying Rates

Not only can the rates vary over time, but the type of the rate provided can vary over time as well.

Example 5.07 In the first year, the annual effective interest rate is 4%. In the second year, the discount rate convertible semiannually is 5%. In the third year, the force of interest is 6%. In the fourth year, the force of interest is:

$$0.02t \qquad \text{where } 3 < t \leq 4$$

\$700 is lent today. What is the accumulated value of the loan at the end of the fourth year?

Solution To calculate the accumulated value, we use a mixture of rates:

$$AV_4 = 700(1.04)\left(1 - \frac{0.05}{2}\right)^{-2} e^{0.06(3-2)} e^{\int_3^4 0.02s ds} = 813.1671 \times e^{0.01(16-9)}$$

$$= 813.1671 \times e^{0.07} = \mathbf{872.1284}$$

5.05 Accumulation Functions

The accumulation function, $a(t)$, is the accumulated value of an investment of 1 that is made at time 0. Suppose that a deposit is made at time t and accumulated to time $(t + k)$. The ratio of the accumulated value of the deposit to its initial value is equal to the ratio of the accumulation function's values at those times:

$$\frac{AV_{t+k}}{AV_t} = \frac{a(t+k)}{a(t)}$$

Example 5.08 The accumulation function is:

$$a(t) = 1 + 0.03t + 0.002t^2$$

A deposit of 1,000 is made at time 5. What is the accumulated value of the deposit at time 7?

Solution The ratio of the accumulation function's values can be used to find the time-7 accumulated value:

$$\frac{AV_7}{AV_5} = \frac{a(7)}{a(5)}$$

$$\frac{AV_7}{1{,}000} = \frac{1 + 0.03 \times 7 + 0.002 \times 7^2}{1 + 0.03 \times 5 + 0.002 \times 5^2}$$

$$AV_7 = \frac{1.308}{1.200} \times 1{,}000$$

$$AV_7 = \mathbf{1{,}090}$$

Not all accumulation functions imply a constant force of interest. The accumulation function in the previous example has a force of interest that varies over time.

Example 5.09 The accumulation function is:

$$a(t) = 1 + 0.03t + 0.002t^2$$

Find an expression for the force of interest at time t.

Solution We can use the accumulation function in place of the accumulated value in our expression for the force of interest because the accumulation function is an accumulated value. The accumulation function is the accumulated value of 1:

$$r_t = \frac{\frac{d\left(AV_t\right)}{dt}}{AV_t} = \frac{a'(t)}{a(t)} = \frac{\mathbf{0.03 + 0.004}\boldsymbol{t}}{\mathbf{1 + 0.03}\boldsymbol{t}\mathbf{ + 0.002}\boldsymbol{t}^{\mathbf{2}}}$$

Given an expression for either the accumulation function or the accumulated value, we can determine the equivalent rate of interest earned over a period of time.

Example 5.10 The accumulation function is:

$$a(t) = 1 + 0.03t + 0.002t^2$$

a. Calculate the equivalent level annual effective interest rate earned from time 5 to time 7.

b. Calculate the equivalent level annual effective discount rate earned from time 5 to time 7.

Solution

a. The equivalent annual effective interest rate is the rate that produces the same growth from time 5 to time 7 as the accumulation function:

$$(1+i)^{7-5} = \frac{a(7)}{a(5)}$$

$$(1+i)^2 = \frac{1 + 0.03 \times 7 + 0.002 \times 7^2}{1 + 0.03 \times 5 + 0.002 \times 5^2}$$

$$(1+i)^2 = 1.09$$

$$i = \mathbf{0.04403}$$

b. The equivalent discount rate is the rate the produces the same growth from time 5 to time 7 as the accumulation function:

$$\frac{1}{(1-d)^{7-5}} = \frac{a(7)}{a(5)}$$

$$(1-d)^{-2} = \frac{1 + 0.03 \times 7 + 0.002 \times 7^2}{1 + 0.03 \times 5 + 0.002 \times 5^2}$$

$$(1-d)^{-2} = 1.09$$

$$d = \mathbf{0.04217}$$

Alternatively, since we solved for the annual effective interest rate above, we can convert it to the annual effective discount rate:

$$1+i = \frac{1}{1-d}$$

$$1.04403 = \frac{1}{1-d}$$

$$d = \mathbf{0.04217}$$

5.06 Questions

Question 5.01

The interest rates over the next 4 years will vary as follows:

For the first 1.5 years, the annual effective rate of interest is 7%.

For the subsequent year, the annual interest rate compounded quarterly is 8%.

For the subsequent 1.5 years, the annual interest rate compounded monthly is 6%.

A payment of $100 is to be received in 4 years. Calculate the present value of the payment.

A 76.30 B 76.40 C 76.47 D 76.65 E 81.01

Question 5.02

The interest rates over the next 4 years will vary as follows:

For the first 1.5 years, the annual effective rate of interest is 7%.

For the subsequent year, the annual interest rate compounded quarterly is 8%.

For the subsequent 1.5 years, the annual interest rate compounded monthly is 6%.

A deposit is made now will be accumulated over the next 4 years. Calculate the equivalent level interest rate earned over the 4 years, expressed as a nominal annual interest rate compounded semiannually.

A 6.76% B 6.82% C 6.88% D 7.00% E 7.12%

Question 5.03

The discount rates over the next 4 years will vary as follows:

For the first 1.5 years, the annual effective rate of discount is 7%.

For the subsequent year, the annual discount rate compounded quarterly is 8%.

For the subsequent 1.5 years, the annual discount rate compounded monthly is 6%.

A payment of $100 is to be received in 4 years. Calculate the present value of the payment.

A 75.40 B 75.59 C 76.30 D 80.26 E 82.10

Question 5.04

The discount rates over the next 4 years will vary as follows:

For the first 1.5 years, the annual effective rate of discount is 7%.

For the subsequent year, the annual discount rate compounded quarterly is 8%.

For the subsequent 1.5 years, the annual discount rate compounded monthly is 6%.

A deposit of $100 is made now, and a deposit of $50 is made at the end of 2 years. Calculate the accumulated value of the payment at the end of four years.

A 176.18 B 189.28 C 189.87 D 190.23 E 190.35

Question 5.05

Velma deposits \$100 into an account that earns an annual effective interest rate of 10% for 2 years and a nominal annual interest rate of 8% compounded semiannually for 3 years.

Daphne deposits \$150 into an account that earns a nominal annual discount rate of 12% compounded monthly for 3 years. For the following two years, Daphne's account earns an annual effective discount rate of d.

At the end of 5 years, the accumulated value in Daphne's account is twice the accumulated value in Velma's account.

Calculate d.

A 8.07% B 16.13% C 18.61% D 19.23% E 58.42%

Question 5.06

The force of interest over the next 4 years will vary as follows:

For the first 1.5 years, the annual force of interest is 7%.

For the subsequent year, the annual force of interest is 8%.

For the subsequent 1.5 years, the annual force of interest is 6%.

A payment of \$100 is to be received in 4 years. Calculate the present value of the payment.

A 75.96 B 76.03 C 76.09 D 76.66 E 77.31

Question 5.07

The force of interest over the next 4 years will vary as follows:

For the first 1.5 years, the annual force of interest is 7%.

For the subsequent year, the annual force of interest is 8%.

For the subsequent 1.5 years, the annual force of interest is 6%.

A deposit of \$100 is made now, and a deposit of \$50 is made at the end of 2 years. Calculate the accumulated value of the payment at the end of four years.

A 175.56 B 178.46 C 180.46 D 188.59 E 189.46

Question 5.08

A deposit of D is made now and accumulates at a simple interest rate of 5% per year. Calculate the force of interest at the following points in time:

a. one month after the deposit

b. one year after the deposit

c. five years after the deposit

d. ten years after the deposit

Question 5.09

A deposit of D is made now and accumulates at a simple discount rate of 5% per year. Calculate the force of interest at the following points in time:

a. one month after the deposit

b. one year after the deposit

c. five years after the deposit

d. ten years after the deposit

Question 5.10

A lender loans \$5,000 now and \$$X$ 2 years from now. The loan is repaid at the end of 8 years with a single payment of \$12,000.

The interest rate on the loan is an annual effective interest rate of 5% for the first 5 years. Thereafter, the interest is charged at a force of interest of:

$$r_t = \frac{1}{2+t} \qquad \text{for } t \geq 5$$

Calculate X.

A 1,434.58 B 1,581.62 C 1,743.74 D 2,256.24 E 6,581.62

Question 5.11

In Fund X, the accumulated value of 1 at any time $t > 0$ is $1 + 0.5t$.

In Fund Y, the accumulated value of 1 at any time $t > 0$ is $1 + 0.5t^2$.

T is the time when the force of interest for Fund X is equal to the force of interest for Fund Y.

Calculate T.

A 0.11 B 0.45 C 0.59 D 0.90 E 1.16

Question 5.12

You are given:

(i) The amount in Fund X at time t is $X(t)$. Fund X accumulates at the following force of interest:

$$r_t = \frac{1}{1+t}$$

(ii) The amount in Fund Y at time t is $Y(t)$. Fund Y accumulates at the following force of interest:

$$r_t = \frac{10t}{1+5t^2}$$

(iii) $X(0) = Y(0)$

(iv) $H(t) = X(t) - Y(t)$

(v) T is the time when $H(t)$ is a maximum.

Calculate T.

A 0.10 B 0.20 C 0.25 D 0.50 E 0.75

Question 5.13

At time 0, 1,000 is deposited into Fund A and also into Fund B. Fund A accumulates at a force of interest of:

$$r_t = \frac{1}{3(1+t)^3}$$

Fund B accumulates at an annual effective interest rate of i.

At the end of 5 years, the accumulated value of Fund A is equal to the accumulated value of Fund B.

Calculate i.

A 3.00% B 3.05% C 3.25% D 3.29% E 3.38%

Question 5.14

You are given:

(i) The force of interest is:

$$r_t = \frac{2t^3 + 6t}{t^4 + 6t^2 + 9}$$

(ii) Fund A accumulates with simple interest at a rate of *i*.

(iii) Fund B accumulates at r_t.

(iv) An amount of 1 is deposited in each of Fund A and Fund B at time 0.

(v) At time 1, the amount in Fund A is equal to the amount in Fund B.

At what time is (Fund A – Fund B) maximized?

A 0.25 B 0.33 C 0.50 D 0.67 E 0.75

Question 5.15

The force of interest, δ_t, is:

$$\delta_t = \begin{cases} 0.05 & 0 < t \le 4 \\ 0.001(t^2 - t) & t > 4 \end{cases}$$

Calculate the accumulated value of at time 8 of $100 that is deposited at time 0.

A 107.75 B 136.27 C 136.30 D 137.94 E 138.45

Question 5.16

The force of interest, δ_t, is:

$$\delta_t = \begin{cases} 0.02 & 0 < t \le 6 \\ 0.002(t^2 + t) & t > 6 \end{cases}$$

Calculate the present value at time 2 of $100 that is payable at time 10.

A 49.33 B 50.67 C 51.34 D 53.44 E 58.67

Question 5.17

Marcia and Jan each put $100 into separate accounts at time $t = 0$, where *t* is measured in years.

Marcia's account earns a constant annual effective interest rate of *K*/36, $K > 0$.

Jan's account earns interest at a force of interest of:

$$\delta_t = \frac{1}{K + 0.20t}$$

At the end of 5 years, the amount in each account is *X*.

Calculate *X*.

A 114.68 B 116.67 C 194.96 D 216.14 E 248.83

Question 5.18

Carlos makes deposits of \$100 at time 0 and X at time 4 into a fund. The fund grows at a force of interest of:

$$\delta_t = \frac{t^3}{2{,}000}$$

The amount of interest earned from time 4 to time 8 is also X.

Calculate X.

A 103.25 B 165.69 C 220.98 D 268.94 E 434.62

Question 5.19

The annual force of interest credited to an account at time t is shown below, where t is measured in years:

$$\delta_t = \frac{t^2}{4 + t^3}$$

A deposit of D is made at time 0.

Calculate the number of years until the account is worth 3 times the initial deposit.

A 2.00 B 2.84 C 3.14 D 4.70 E 4.82

Question 5.20

The annual force of interest credited to an account at time t is shown below, where t is measured in years:

$$\delta_t = \frac{\dfrac{t^2}{120}}{4 + \dfrac{t^3}{180}}$$

A payment of \$800 will be received at time 5.

Calculate the present value of the payment.

A 369.23 B 413.45 C 580.82 D 681.66 E 738.46

Question 5.21

Brian deposits \$100 into a fund today.

Interest for the first year is credited at a nominal discount rate of 6% compounded monthly.

Interest for the second year is credited at a nominal interest rate of 7% compounded quarterly.

Interest for the third year is credited at a constant annual force of interest of 10%.

Calculate the accumulated value at the end of 3 years.

A 111.54 B 124.68 C 125.21 D 125.77 E 125.80

Question 5.22

Steve will receive \$100 in 3 years. Over the next three years, the interest rates will vary as follows:

In the first year, the nominal discount rate is 6% compounded monthly.

In the second year, the nominal interest rate is 7% compounded quarterly.

In the third year, the constant annual force of interest is 10%.

Calculate the present value at time 0 of the \$100 payment.

A 79.49 B 79.86 C 80.21 D 84.38 E 89.65

Question 5.23

A deposit of 500 is made into a fund today.

For the first 4 years, the fund earns an annual effective discount rate of d.

For the next 2 years, the fund earns a nominal interest rate of d compounded semiannually.

At the end of 6 years, the accumulated value in the fund is 767.

Calculate d.

A 5.34% B 6.92% C 7.00% D 18.42% E 26.25%

Question 5.24

Norma deposits 100 into a fund today and 50 ten years later.

Interest for the first 7 years is credited at a nominal discount rate of d compounded semiannually, and thereafter at a nominal interest rate of 7% compounded monthly.

The accumulated balance in the fund at the end of twenty years is 500.

Calculate d.

A 1.68% B 3.35% C 6.71% D 13.41% E 15.10%

Question 5.25

The accumulation function is a function of the number of years elapsed, t:

$$a(t) = 1 + 0.04t + 0.002t^2$$

Calculate the equivalent level annual effective discount rate earned during the first 15 years.

A 4.5% B 4.6% C 4.7% D 4.8% E 4.9%

Chapter 6: Level Annuities Payable Once per Time Unit

6.01 Geometric Series

A geometric series is a sum of terms, where the ratio of each term to its predecessor is a constant. The geometric series below has n terms, the first of which is a, and the ratio is r:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

In this section, the ratio r isn't the force of interest. We usually try to avoid using the same variable in two different ways, but in this case r is the best choice for both the force of interest and the ratio in a geometric series.

Let's multiply both sides by r:

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

We subtract the second equation from the first to obtain a new equation:

$$\begin{aligned} S_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ \underline{rS_n} &\underline{= ar + ar^2 + ar^3 + \cdots + ar^n} \\ S_n - rS_n &= a - ar^n \end{aligned}$$

Now we can solve for S_n:

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n)$$

$$S_n = \frac{a(1-r^n)}{1-r}$$

We can think of ar^n as the term that would come next if the geometric series were to continue for one additional term.

Sum of a Geometric Series

6.01 The sum of a geometric series with n terms is:

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{\text{First term} - \text{Term that would come next}}{1 - \text{Ratio}}$$

where:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

If $-1 < r < 1$, then:

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$$

Therefore, the sum of an infinite series can be written as:

$$a + ar + ar^2 + \cdots = \frac{a}{1-r} \qquad \text{where: } -1 < r < 1$$

6.02 Annuity-Immediate

An **annuity-immediate** consists of a series of payments that occur at the end of each **payment period**. The payment period is the time interval between payments, and in this chapter the payment period is equal to one unit of time. The number of units of time over which an annuity is paid is the **term** of the annuity.

Suppose that an annuity makes a payment of 1 at the end of each of n units of time. The payment period is therefore equal to one unit of time and the term of the annuity is equal to n:

Payment		1	1	1	...	1
Time	0	1	2	3	...	n

The present value of this stream of payments is the sum of a geometric series:

$$
\begin{aligned}
PV_0 &= \frac{1}{1+i} + \left(\frac{1}{1+i}\right)^2 + \left(\frac{1}{1+i}\right)^3 + \cdots + \left(\frac{1}{1+i}\right)^n \\
&= v + v^2 + v^3 + \cdots + v^n = \frac{v - v^{n+1}}{1-v} = \frac{1+i}{1+i} \times \frac{v - v^{n+1}}{1-v} = \frac{1-v^n}{1+i-1} \\
&= \frac{1-v^n}{i}
\end{aligned}
$$

The accumulated value at time n is also the sum of a geometric series:

$$
\begin{aligned}
AV_n &= (1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \cdots + 1 \\
&= 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} \\
&= \frac{1-(1+i)^n}{1-(1+i)} = \frac{1-(1+i)^n}{-i} \\
&= \frac{(1+i)^n - 1}{i}
\end{aligned}
$$

There is another way to derive the formula for the accumulated value. As noted in Section 3.08, under compound interest, equations of value hold across time.

This means that we can begin with the time-0 equation of value and convert it into a time-n equation of value by accumulating both sides of the equation for n years:

$$PV_0 = \frac{1-v^n}{i}$$

$$(1+i)^n PV_0 = (1+i)^n \frac{1-v^n}{i}$$

$$AV_n = \frac{(1+i)^n - 1}{i}$$

The Key Concept below introduces the following notation:

$$a_{\overline{n}|i} = \frac{1}{1+i} + \left(\frac{1}{1+i}\right)^2 + \left(\frac{1}{1+i}\right)^3 + \cdots + \left(\frac{1}{1+i}\right)^n$$

$$= \begin{cases} \text{Present value of an annuity that pays 1 at} \\ \text{the end of each of } n \text{ units of time, where } i \text{ is} \\ \text{the effective rate of interest per unit of time} \end{cases}$$

$$s_{\overline{n}|i} = (1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \cdots + 1$$

$$= \begin{cases} \text{Accumulated value of an annuity that pays 1 at} \\ \text{the end of each of } n \text{ units of time, where } i \text{ is} \\ \text{the effective rate of interest per unit of time} \end{cases}$$

6.02

Annuity-Immediate

An annuity-immediate that pays 1 at the end of each time unit for n units of time has the following present value and accumulated value:

$$a_{\overline{n}|i} = \frac{1-v^n}{i} \qquad\qquad s_{\overline{n}|i} = \frac{(1+i)^n - 1}{i}$$

where:

$i =$ Effective interest rate per unit of time

When the effective interest rate per unit of time is not in question, it is common to drop the i from the notation, so the present value and the accumulated values of an annuity-immediate are often written as:

$$a_{\overline{n}|} \qquad \& \qquad s_{\overline{n}|}$$

Example 6.01

The annual effective interest rate is 4%. An annuity-immediate makes payments of \$5 at the end of each year for 8 years. Find the present value of the annuity-immediate.

Solution

The present value is:

$$PV_0 = 5v + 5v^2 + 5v^3 + \cdots + 5v^8 = 5a_{\overline{8}|0.04} = 5 \times \frac{1-v^8}{0.04} = 5 \times \frac{1-(1.04)^{-8}}{0.04}$$

$$= 5 \times 6.7327 = \mathbf{33.66}$$

We can also find this value using the BA II Plus calculator:

8 [*N*] 4 [*I/Y*] 5 [*PMT*] [CPT] [*PV*]

PV = –33.66 Answer = **33.66**

If you used the BA II Plus but didn't obtain a result of –33.66, then you may want to check your calculator's settings, which are discussed in Section 1.01.

The interest rate of 4% in the example above is entered into the BA II Plus as 4, not 0.04.

The BA II Plus can solve for any one of the following if the other 4 are provided: *N*, *I/Y*, *PV*, *PMT*, and *FV*.

The table below shows how the BA II Plus defines the variables and also shows how we use them in this text.

	BA II Plus	**This Text**
N	Number of periods	Number of payments, *n*
I/Y	Interest rate per year	Interest rate per unit of time, *i*
PV	Present value	Present value, PV_0
PMT	Payment	Payment
FV	Future value	Future value, FV_n

In the example above, the worksheet register was assumed to be cleared, so that *FV* = 0. Therefore, the calculator has values for *N*, *I/Y*, *PMT*, and *FV*, and when we enter [CPT] [*PV*], the calculator calculates the present value that satisfies the following expression:

$$PV + Pmt \times a_{\overline{N|}I/Y} + FV \times v^N = 0$$

Using this text's notation, the expression is:

$$PV_0 + Pmt \times a_{\overline{n|}i} + FV_n \times v^n = 0$$

With *FV* = 0, this simplifies to:

$$PV_0 = -Pmt \times a_{\overline{n|}i}$$

Therefore, the calculator's result for *PV* is the negative of the present value of the payments. This is shown in the example above as:

$PV = -33.66$

Since the calculator's result is the negative of the present value, we multiply by -1 to obtain the answer, which is shown in the example above as:

Answer = 33.66

Multiplying the calculator's answer by –1 might seem confusing, but if you keep the following equivalent equations in mind, then the result from the calculator is easy to interpret:

Equation of value at time 0: $PV + Pmt \times a_{\overline{N|}I/Y} + FV \times v^N = 0$

Equation of value at time *n*: $PV(1 + I/Y)^N + Pmt \times s_{\overline{N|}I/Y} + FV = 0$

The second equation above is just the first equation multiplied by $(1 + I/Y)^N$.

Example 6.02 The annual effective interest rate is 4%. An annuity-immediate makes level payments at the end of each year for 8 years. The present value of the annuity-immediate 33.6637. Calculate the amount of each level payment.

Solution We solve for level payment, *PMT*, below:

$$PV_0 = PMT \times a_{\overline{n|}i}$$

$$33.6637 = PMT \times a_{\overline{8|}0.04}$$

$$33.6637 = PMT \times \frac{1-(1.04)^{-8}}{0.04}$$

$$33.6637 = PMT \times 6.7327$$

$$PMT = \mathbf{5.00}$$

We can also find this value using the BA II Plus calculator:

8 [*N*] 4 [*I/Y*] 33.6637 [*PV*] [CPT] [*PMT*]

PMT = –5.00 Answer = **5.00**

Example 6.03 The annual effective interest rate is 4%. An annuity-immediate makes payments of \$5 at the end of each year for 8 years. Calculate the accumulated value of the annuity-immediate at the end of 8 years.

Solution The accumulated value is:

$$AV_8 = 5(1.04)^7 + 5(1.04)^6 + 5(1.04)^5 + \cdots + 5(1.04) + 5 = 5s_{\overline{8}|0.04}$$

$$= 5 \times \frac{1.04^8 - 1}{0.04} = 5 \times 9.2142 = \mathbf{46.07}$$

We can also find this value using the BA II Plus calculator:

8 [*N*] 4 [*I/Y*] 5 [*PMT*] [CPT] [*FV*]

FV = –46.07 Answer = **46.07**

Since much of the information pertaining to the example above also pertains to the example preceding it, it is not necessary to clear the TVM register when going from Example 6.02 to Example 6.03. Instead, we can leave N = 8 and I/Y = 4 and then set PV to 0 with 0 [PV].

A perpetuity-immediate is an annuity-immediate that pays out forever. The present value of a perpetuity-immediate that pays 1 at the end of each time unit forever is:

$$a_{\overline{\infty}|i} = \lim_{n \to \infty} a_{\overline{n}|i} = \lim_{n \to \infty} \frac{1 - v^n}{i} = \frac{1}{i}$$

Example 6.04 The annual effective interest rate is 6%. A perpetuity-immediate makes payments of \$5 at the end of each year forever. Calculate the present value of the perpetuity-immediate.

Solution The present value is:

$$PV_0 = 5a_{\overline{\infty}|0.06} = 5 \times \frac{1}{0.06} = \mathbf{83.33}$$

6.03 Annuity-Due

An **annuity-due** is similar to an annuity-immediate, but it consists of payments that are made at the *beginning* of each payment period.

Suppose that an annuity makes a payment of 1 at the beginning of each unit of time for n units of time. The payment period is therefore equal to one unit of time and the term of the annuity is equal to n:

Payment	1	1	1	...	1	
Time	0	1	2	...	$n-1$	n

Even though the annuity makes its final payment at time $(n-1)$, we still say that the term of the annuity is n. This is because the annuity is considered to be making payments for n years; even though the payments are made at the beginning of each year. That is, the term of an annuity is the number of units of time (usually years) over which the annuity is paid, regardless of when the payments are made within each unit of time.

The present value of this stream of payments is the sum of a geometric series:

$$PV_0 = 1 + \frac{1}{1+i} + \left(\frac{1}{1+i}\right)^2 + \cdots + \left(\frac{1}{1+i}\right)^{n-1} = 1 + v + v^2 + \cdots + v^{n-1}$$

$$= \frac{1-v^n}{1-v} = \frac{1-v^n}{\frac{1+i}{1+i} - \frac{1}{1+i}} = \frac{1-v^n}{\frac{i}{1+i}} = \frac{1-v^n}{d}$$

We use an umlaut to differentiate an annuity-due from an annuity-immediate:

$$\ddot{a}_{\overline{n}|i} = \frac{1-v^n}{d}$$

We can derive the same formula for an annuity-due using the formula for an annuity-immediate:

$$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + \cdots v^{n-1} = (1+i)\left[v + v^2 + v^3 + \cdots v^n\right] = (1+i)a_{\overline{n}|i}$$

$$= (1+i)\frac{1-v^n}{i} = \frac{1-v^n}{\frac{i}{1+i}} = \frac{1-v^n}{d}$$

Let's accumulate both sides of the time-0 equation of value to obtain the time-n equation of value:

$$PV_0 = \frac{1-v^n}{d}$$

$$(1+i)^n PV_0 = (1+i)^n \frac{1-v^n}{d}$$

$$AV_n = \frac{(1+i)^n - 1}{d}$$

$$\ddot{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d}$$

We can derive the same formula by recognizing that the accumulated value is a geometric series:

$$\ddot{s}_{\overline{n}|i} = (1+i)^n + (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) = \frac{(1+i)^n - 1}{1-v} = \frac{(1+i)^n - 1}{1 - \frac{1}{1+i}}$$

$$= \frac{(1+i)^n - 1}{\frac{1+i}{1+i} - \frac{1}{1+i}} = \frac{(1+i)^n - 1}{\frac{i}{1+i}} = \frac{(1+i)^n - 1}{d}$$

As with the present value of the annuity-due, we use an umlaut to differentiate the accumulated value from that of an annuity-immediate.

6.03

Annuity-Due

An annuity-due that pays 1 at the beginning of each time unit for n units of time has the following present value and accumulated value:

$$\ddot{a}_{\overline{n}|i} = \frac{1-v^n}{d} \qquad \ddot{s}_{\overline{n}|i} = \frac{(1+i)^n - 1}{d}$$

where:

i = Effective interest rate per unit of time

When the effective interest rate per unit of time is not in question, it is common to drop the i from the notation so that the present value and the accumulated values of an annuity-due can be written as:

$$\ddot{a}_{\overline{n}|} \qquad \& \qquad \ddot{s}_{\overline{n}|}$$

Since each of the payments from an annuity-due is made one unit of time earlier than the payments from an otherwise equivalent annuity-immediate, we have:

$$\ddot{a}_{\overline{n}|} = (1+i)a_{\overline{n}|}$$
$$\ddot{s}_{\overline{n}|} = (1+i)s_{\overline{n}|}$$

Example 6.05

Determine whether the expressions below are true or false:

a. $a_{\overline{n}|} = v \times \ddot{a}_{\overline{n}|}$

b. $\ddot{a}_{\overline{n+1}|} = a_{\overline{n}|} + 1$

c. $\ddot{s}_{\overline{n}|} = (1+i)^{n+1}a_{\overline{n}|}$

d. $\ddot{a}_{\overline{n}|}(1+v^n) = \ddot{a}_{\overline{2n}|}$

e. $\dfrac{\ddot{s}_{\overline{2n}|}}{\ddot{s}_{\overline{n}|}} - 2 = i \times s_{\overline{n}|}$

f. $s_{\overline{n}|} = \ddot{s}_{\overline{n-1}|} + 1$

Solution

All of the expressions are **true**:

a. $a_{\overline{n}|} = v + v^2 + \cdots + v^n = v(1 + v + \cdots + v^{n-1}) = v \times \ddot{a}_{\overline{n}|}$

b. $\ddot{a}_{\overline{n+1}|} = 1 + v + \cdots + v^n = 1 + a_{\overline{n}|} = a_{\overline{n}|} + 1$

c. $\ddot{s}_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + \cdots + (1+i) = (1+i)^n\left[1 + v + \cdots + v^{n-1}\right] = (1+i)^n\ddot{a}_{\overline{n}|}$

$$= (1+i)^n(1+i)a_{\overline{n}|} = (1+i)^{n+1}a_{\overline{n}|}$$

d. $\ddot{a}_{\overline{n}|}(1+v^n) = (1 + v + v^2 + \cdots + v^{n-1})(1+v^n)$

$$= (1 + v + v^2 + \cdots + v^{n-1}) + (v^n + v^{n+1} + v^{n+2} + \cdots + v^{2n-1}) = \ddot{a}_{\overline{2n}|}$$

e. $\dfrac{\ddot{s}_{\overline{2n}|}}{\ddot{s}_{\overline{n}|}} - 2 = \dfrac{\dfrac{(1+i)^{2n}-1}{d}}{\dfrac{(1+i)^n-1}{d}} - 2 = \dfrac{(1+i)^{2n}-1}{(1+i)^n-1} - 2 = \dfrac{\left[(1+i)^n-1\right]\left[(1+i)^n+1\right]}{(1+i)^n-1} - 2$

$$= (1+i)^n + 1 - 2 = (1+i)^n - 1 = i \times \frac{(1+i)^n-1}{i} = i \times s_{\overline{n}|}$$

f. $s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + (1+i)^{n-3} + \cdots + (1+i) + 1 = \ddot{s}_{\overline{n-1}|} + 1$

Example 6.06

The annual effective interest rate is 4%. An annuity-due makes payments of \$5 at the beginning of each year for 8 years. Find the present value of the annuity-due.

Solution

The present value is:

$$PV_0 = 5 + 5v + 5v^2 + \cdots + 5v^7 = 5\ddot{a}_{\overline{8}|0.04} = 5 \times \frac{1-v^8}{\frac{0.04}{1.04}} = 5 \times \frac{1-(1.04)^{-8}}{\frac{0.04}{1.04}}$$

$$= 5 \times 7.0021 = \mathbf{35.01}$$

We can also find this value using the BA II Plus calculator. Let's consider two ways to use the calculator.

1. First, we find the present value of a corresponding annuity-immediate:

   8 [*N*] 4 [*I/Y*] 5 [*PMT*] [CPT] [*PV*]

   PV = −33.66 Answer = 33.66

Next, we multiply the present value of the annuity-immediate by 1.04 to find the present value of the annuity-due:

$$33.66 \times (1+i) = 33.66 \times (1.04) = \mathbf{35.01}$$

2. Second, we set the calculator to treat the payments as occurring at the beginning of each year by setting it to the BGN mode:

   [2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

Next we compute the present value of the annuity-due:

   8 [*N*] 4 [*I/Y*] 5 [*PMT*] [CPT] [*PV*]

   PV = −35.01 Answer = **35.01**

If you toggled the calculator to the BGN mode, then we suggest that you either immediately toggle the calculator back to the END mode or make it a habit to always check the calculator mode before using the TVM worksheet. If you forget which mode the calculator is in, it is easy to answer questions incorrectly!

When the calculator is set to the BGN mode, it solves one of the following when it is given the other 4: *N*, *I/Y*, *PV*, *PMT*, and *FV*. The calculator calculates the missing value that satisfies the following two equivalent expressions:

Equation of value at time 0: $PV + Pmt \times \ddot{a}_{\overline{N}|I/Y} + FV \times v^N = 0$

Equation of value at time *n*: $PV(1+I/Y)^N + Pmt \times \ddot{s}_{\overline{N}|I/Y} + FV = 0$

The second equation above is just the first equation multiplied by $(1+I/Y)^N$.

A perpetuity-due is an annuity-due that pays out forever. The present value of a perpetuity-due that pays 1 at the beginning of each time unit forever is:

$$\ddot{a}_{\overline{\infty}|i} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|i} = \lim_{n\to\infty} \frac{1-v^n}{d} = \frac{1}{d}$$

Another way to find the value of a perpetuity-due is to observe that a payment now plus payments from a perpetuity-immediate would result in the same payments as would be received from a perpetuity-due. This suggests that 1 plus the value of a perpetuity-immediate is equal to the value of a perpetuity-due. This is verified below:

$$\ddot{a}_{\overline{\infty}|i} = \frac{1}{d} = \frac{1+i}{i} = \frac{1}{i} + 1 = a_{\overline{\infty}|i} + 1$$

Example 6.07 The annual effective interest rate is 6%. A perpetuity-due makes payments of $5 at the beginning of each year forever. Calculate the present value of the Perpetuity-due.

Solution The present value is:

$$PV_0 = 5\ddot{a}_{\overline{\infty}|0.06} = 5 \times \frac{1}{d} = 5 \times \frac{1}{\frac{0.06}{1.06}} = \mathbf{88.33}$$

6.04 Deferred Annuity

A **deferred annuity** is an annuity that does not commence until some amount of time has elapsed. We use *k* to denote the length of the deferral.

Deferred Annuity-Immediate

Consider a deferred annuity-immediate that makes payments of 1 at the end of each unit of time, for n units of time, commencing after k units of time have elapsed:

Payment					1	1	1	...	1	1
Time	0	1	...	k	$k+1$	$k+2$	$k+3$	...	$k+n-1$	$k+n$

No payment is made at time k because the first payment occurs at the end of the first unit of time after k, which means that the first payment occurs at time $k+1$.

At time k, this annuity will be a regular annuity-immediate, and its value will be:

$$PV_k = a_{\overline{n}|}$$

To find the value at time 0, we discount from time k to time 0:

$$PV_0 = v^k \times PV_k = v^k \times a_{\overline{n}|}$$

To indicate the present value of an annuity that is deferred for k units of time, a k followed by a horizontal bar is placed in front of the annuity symbol:

$$_{k|}a_{\overline{n}|} = v^k \times a_{\overline{n}|}$$

6.04

Deferred Annuity-Immediate Present Value

The present value of a deferred annuity-immediate that commences after k units of time and then pays 1 at the end of each unit of time for n units of time is:

$$_{k|}a_{\overline{n}|} = v^k \times a_{\overline{n}|}$$

Example 6.08 The annual effective interest rate is 6%. A deferred annuity-immediate makes payments of \$1,000 per year for 15 years, after a deferral period of 5 years. Calculate the present value of the deferred annuity-immediate.

Solution The present value is:

$$PV_0 = 1{,}000 \times {}_{5|}a_{\overline{15}|} = 1{,}000 \times v^5 \times a_{\overline{15}|} = 1{,}000 \times \frac{1}{1.06^5} \times \frac{1-v^{15}}{0.06}$$

$$= 1{,}000 \times 0.7473 \times 9.7122 = \mathbf{7{,}257.56}$$

We can also find this value using the BA II Plus calculator. First we obtain the present value of a regular 15-year annuity immediate that pays \$1,000 per year:

15 [*N*] 6 [*I/Y*] 1,000 [*PMT*] [CPT] [*PV*]

PV = −9,712.25

Now enter this value as a future value, put zero in for the present value and payments, and then discount the −9,712.25 back 5 years:

[*FV*] 0 [*PV*] [*PMT*] 5 [*N*] [CPT] [*PV*]

PV = **7,257.56**

The next two examples use one month as the unit of time.

Example 6.09 The nominal annual interest rate compounded monthly is 9%. A deferred annuity-immediate makes payments of \$100 per month for 10 years, after a deferral period of 3 years. Calculate the present value of the deferred annuity-immediate.

Solution Since this annuity pays monthly, let's use a time unit of 1 month. The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.09}{12} = 0.0075$$

Measured in months, the deferral period of 3 years is 36 months:

$$k = 36$$

There are 120 monthly payments:

$$n = 120$$

The present value is:

$$PV_0 = 100 \times {}_{36|}a_{\overline{120}|0.0075} = 100 \times v^{36} \times a_{\overline{120}|0.0075} = 100 \times \frac{1}{1.0075^{36}} \times \frac{1 - v^{120}}{0.0075}$$

$$= 100 \times 0.7641 \times 78.9417 = \mathbf{6{,}032.32}$$

We can also find this value using the BA II Plus calculator. First we obtain the present value of a regular 120-month annuity-immediate that pays 100 per month:

120 [*N*] 0.75 [*I/Y*] 100 [*PMT*] [CPT] [*PV*]

PV = −7,894.17

Now enter this value as a future value, put zero in for the present value and payments, and then discount the −7,894.17 back 36 months:

[*FV*] 0 [*PV*] [*PMT*] 36 [*N*] [CPT] [*PV*]

PV = **6,032.32**

The present value of a deferred perpetuity-immediate can be found by taking the limit as *n* goes to infinity:

$${}_{k|}a_{\overline{\infty}|} = \lim_{n\to\infty} {}_{k|}a_{\overline{n}|} = \lim_{n\to\infty} v^k \times a_{\overline{n}|} = v^k \times \frac{1}{i}$$

Example 6.10 The nominal annual interest rate compounded monthly is 9%. A deferred perpetuity-immediate makes payments of \$100 per month forever, after a deferral period of 3 years. Calculate the present value of the deferred perpetuity-immediate.

Solution Since this annuity pays monthly, let's use a time unit of 1 month. The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.09}{12} = 0.0075$$

Measured in months, the deferral period of 3 years is 36 months:

$$k = 36$$

The present value is:

$$PV_0 = 100 \times {}_{36|}a_{\overline{\infty}|0.0075} = 100 \times v^{36} \times a_{\overline{\infty}|0.0075} = 100 \times \frac{1}{1.0075^{36}} \times \frac{1}{0.0075}$$

$$= 100 \times 0.7641 \times 133.3333 = \mathbf{10{,}188.65}$$

Deferred Annuity-Due

Consider a deferred annuity-due that makes payments at the beginning of each unit of time, for n units of time, commencing after k units of time have elapsed:

Payment				1	1	1	1	...	1	
Time	0	1	...	k	$k+1$	$k+2$	$k+3$	...	$k+n-1$	$k+n$

The first payment is made at time k.

At time k, this annuity will be a regular annuity-due, and its value will be:

$$PV_k = \ddot{a}_{\overline{n}|}$$

To find the value at time 0, we discount from time k to time 0:

$$PV_0 = v^k \times PV_k = v^k \times \ddot{a}_{\overline{n}|}$$

As before, the deferral period is indicated by a k followed by a horizontal bar:

$${}_{k|}\ddot{a}_{\overline{n}|} = v^k \times \ddot{a}_{\overline{n}|}$$

6.05

Deferred Annuity-Due Present Value

The present value of a deferred annuity-due that commences after k units of time and then pays 1 at the beginning of each unit of time for n units of time is:

$${}_{k|}\ddot{a}_{\overline{n}|} = v^k \times \ddot{a}_{\overline{n}|}$$

Why haven't we provided formulas for the accumulated values of deferred annuities?

Because the time interval over which each payment is accrued is not affected by the deferral period.

We could write the following expressions for the accumulated values at time n for an annuity-immediate and an annuity-due, but since they are the same as the regular formulas, there is no need for them:

$${}_{k|}s_{\overline{n}|} = s_{\overline{n}|}$$

$${}_{k|}\ddot{s}_{\overline{n}|} = \ddot{s}_{\overline{n}|}$$

Example 6.11

The nominal interest rate convertible quarterly is 8%. A deferred annuity-due makes payments of \$300 every 3 months for 15 years, after a deferral period of 5 years. Calculate the present value of the deferred annuity-immediate.

Solution

Since this annuity pays quarterly, let's use a time unit of 3 months. The quarterly effective interest rate is:

$$\frac{i^{(4)}}{4} = \frac{0.08}{4} = 0.02$$

Measured in quarters, the deferral period of 5 years is 20 quarters:

$$k = 5 \times 4 = 20$$

There are 60 quarterly payments:

$$n = 15 \times 4 = 60$$

The present value is:

$$PV_0 = 300 \times {}_{20|}\ddot{a}_{\overline{60}|0.02} = 300 \times v^{20} \times \ddot{a}_{\overline{60}|0.02} = 300 \times \frac{1}{1.02^{20}} \times \frac{1 - v^{60}}{0.02} \times 1.02$$

$$= 300 \times 0.6730 \times 35.4561 = \mathbf{7{,}158.28}$$

We can also find this value using the BA II Plus calculator. First, set the calculator to the BGN mode:

[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]

We obtain the present value of a regular 15-year annuity-due that pays $300 per quarter:

60 [*N*] 2 [*I/Y*] 300 [*PMT*] [CPT] [*PV*]

PV = −10,636.83

Now enter this value as a future value, put zero in for the present value and payments, and then discount the −9,712.25 back for 20 quarters:

[*FV*] 0 [*PV*] [*PMT*] 20 [*N*] [CPT] [*PV*]

PV = **7,158.28**

If you prefer to avoid the BGN mode of the calculator, then you can find the present value of an annuity-immediate in the example above and then multiply by 1.02 to convert the present value of the annuity-immediate into the present value of the corresponding annuity-due. With the calculator set to the END mode, we can find the present value of a regular 15-year annuity-due that pays $300 per quarter:

60 [N] 2 [I/Y] 300 [PMT] [CPT] [PV]

PV = −10,428.27

[×] 1.02 [=] −10,636.83

The present value of a deferred perpetuity-due can be found by taking the limit as n goes to infinity:

$${}_{k|}\ddot{a}_{\overline{\infty}|} = \lim_{n\to\infty} {}_{k|}\ddot{a}_{\overline{n}|} = \lim_{n\to\infty} v^k \times \ddot{a}_{\overline{n}|} = v^k \times \frac{1}{d}$$

Example 6.12 The nominal annual interest rate compounded monthly is 9%. A deferred perpetuity-due make payments of $100 per month forever, after a deferral period of 3 years. Calculate the present value of the deferred perpetuity-due.

Solution Since this annuity pays monthly, let's use a time unit of 1 month. The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.09}{12} = 0.0075$$

Measured in months, the deferral period of 3 years is 36 months:

$$k = 36$$

The present value is:

$$PV_0 = 100 \times {}_{36|}\ddot{a}_{\overline{\infty}|0.0075} = 100 \times v^{36} \times \ddot{a}_{\overline{\infty}|0.0075} = 100 \times \frac{1}{1.0075^{36}} \times \frac{1}{\frac{0.0075}{1.0075}}$$

$$= 100 \times 0.7641 \times 134.3333 = \mathbf{10{,}265.07}$$

6.05 Questions

Question 6.01

Greg makes deposits of 1,000 at the end of each year for n years into a fund.

At time n, he uses the accumulated value of the fund to purchase an annuity immediate that makes payments of 2,061.34 at the end of each year for ten years.

The annual effective interest rate is 5%.

Calculate n.

A 8 B 9 C 10 D 11 E 12

Question 6.02

Sheila will receive payments of \$2,000 at the end of each year for 15 years. Sue will receive payments of X at the end of each year for 5 years.

At an annual effective interest rate of 8%, the present value of Sheila's payments is equal to the present value of Sue's payments.

Calculate X.

A 3,969.96 B 4,287.55 C 4,630.56 D 5,001.00 E 6,832.59

Question 6.03

Linda makes deposits of X at the end of each year for five years.

Ten years after making the final deposit, Linda purchases an annuity-immediate that pays 10,000 at the end of every year for 30 years.

The annual effective interest rate is 8%. Calculate X.

A 8,888.51 B 13,060.13 C 19,189.62 D 41,428.95 E 89,441.99

Question 6.04

The purchase price of a new car is X. Haley purchases the car with a down payment of 5,000 and an agreement pay 500 per month at the end of each month for five years. The annual nominal interest rate is 7% compounded monthly.

Calculate X.

A 20,251 B 25,251 C 29,601 D 30,251 E 30,381

Question 6.05

Hector will receive 1,700 at the end of each year for 5 years.

Jill will receive 1,000 at the end of each year for 10 years.

At an annual effective interest rate of i, the present values of the two annuities are equal.

Calculate i.

A 6.89% B 7.39% C 10.07% D 11.20% E 16.81%

Question 6.06

A perpetuity-immediate pays 100 at the end of each year.

An annuity-immediate pays 114.16 at the end of each year for 20 years.

The present value of the perpetuity-immediate is equal to the present value of the annuity-immediate.

Calculate the annual effective interest rate.

A 6.6% B 9.9% C 11.0% D 12.4% E 14.2%

Question 6.07

You are given:

$$a_{\overline{3n}|} = 26.0000 \quad \text{and} \quad a_{\overline{6n}|} = 39.3119$$

Determine $a_{\overline{9n}|}$.

A 42.80 B 46.13 C 52.32 D 52.62 E 60.11

Question 6.08

Amy purchased a car on January 1, 2016 with a $15,000 loan. Her loan carries an interest rate of 22% per year convertible monthly.

She pays $800 per month starting February 1, 2016, and she makes an extra payment of $1,200 on June 1, 2016. Her last payment will be a partial payment.

Determine when she makes the last full payment of $800.

A August 1, 2017

B September 1, 2017

C October 1, 2017

D November 1, 2017

E December 1, 2017

Question 6.09

Which of the following expressions does NOT represent a valid expression for $a_{\overline{n}|}$?

A $\dfrac{v^n - 1}{(v-1)(1+i)}$

B $\dfrac{(1+i)^n - 1}{1-v}v^n$

C $s_{\overline{n}|} \times (1 - iv)^n$

D $\left(\dfrac{1-v^n}{iv}\right)\left(\dfrac{1}{1+i}\right)$

E $\dfrac{1 + v + v^2 + \cdots + v^{n-1}}{1+i}$

Question 6.10

To accumulate 36,419.74 at the end of $3n$ years, deposits of 174 are made at the end of the first n years, and deposits of 87 are made at the end of each of the next $2n$ years.

The annual effective interest rate is i, and you are given that:

$$(1+i)^n = 3$$

Calculate i.

A 7.64% B 10.51% C 10.75% D 14.81% E 15.05%

Question 6.11

An investor wishes to accumulate 8,000 in a fund at the end of 17 years. To accomplish this, she plans to make equal deposits of X at the end of each year for the first 7 years. The fund earns an annual effective rate of 4% for the first 7 years and 8% for the next 10 years.

Calculate X.

A 338 B 415 C 469 D 684 E 1,013

Question 6.12

The present values of the following three annuities are equal:

(i) A perpetuity-immediate paying 1 each year, calculated at an annual effective rate of 8%

(ii) A 40-year annuity-immediate paying 1 each year, calculated at an annual effective rate of i%.

(iii) An n –year annuity-immediate paying 1 each year, calculated at an annual effective rate of $(i - 1)$%

Calculate n.

A 27 B 30 C 32 D 39 E 40

Question 6.13

An annuity-immediate makes payments of \$10 per year for 10 years.

An annuity-due that makes 12 annual payments of X has the same present value as the annuity-immediate.

The annual effective interest rate is 8%.

Calculate X.

A 7.07 B 7.63 C 8.24 D 8.90 E 9.62

Question 6.14

Deposits of 1 per year are made at the beginning of each year for 10 years. The annual effective interest rate is 4%.

Calculate the accumulated value on the date of the last deposit.

A 11.01 B 11.54 C 12.01 D 12.48 E 12.96

Question 6.15

Sheila will receive payments of \$2,000 at the beginning of each year for 15 years. Sue will receive payments of X at the beginning of each year for 5 years.

At an annual effective interest rate of 8%, the present value of Sheila's payments is equal to the present value of Sue's payments.

Calculate X.

A 3,969.96 B 4,287.55 C 4,630.56 D 5,001.00 E 6,832.59

Question 6.16

A perpetuity-immediate makes payments of 3 per year, with the first payment occurring one year from now.

A perpetuity-due makes payments of X per year, with first payment occurring now.

The present values of the two perpetuities are the same. The annual effective discount rate is 5%.

Calculate X.

A 2.82 B 2.83 C 2.84 D 2.85 E 2.86

Question 6.17

Regarding the statements below, you are given:

- The interest rate is greater than zero.
- X is greater than zero.
- The annuities and perpetuities begin now.
- The annuities and perpetuities make annual payments.

Consider the following statements:

I. The present value of a perpetuity-due paying X per year is greater than the present value of perpetuity-immediate paying X per year.

II. The present value of a perpetuity-immediate paying X per year is greater than the present value of an annuity-immediate paying X per year.

III. The present value of a perpetuity-immediate paying X per year is greater than the present value of an annuity-due paying X per year.

Which of the statements above are TRUE?

A I only
B II only
C III only
D I and II only
E I and III only

Question 6.18

An annuity pays 1 at the end of each year for n years. Using an annual effective interest rate of i, the accumulated value of the annuity at time $(n + 1)$ is 30.7725. It is known that $(1+i)^n = 3.7975$.

Calculate n.

A 12 B 13 C 14 D 15 E 16

Question 6.19

At the birth of their son, new parents decide to provide $40,000 at each of their son's 19th, 20th, 21st, and 22nd birthdays to fund his college education. They plan to contribute X at each of their son's 1st through 18th birthdays to fund the four $40,000 withdrawals.

The parents anticipate earning a constant 4% annual effective interest rate on their contributions.

Given that $v = 1/1.04$, determine which of the following equations of value can be used to calculate X.

A $\quad X\sum_{k=1}^{18} v^k = 40{,}000\left[1+v+v^2+v^3\right]$

B $\quad X\sum_{k=0}^{18} 1.04^k = 40{,}000\left[v+v^2+v^3+v^4\right]$

C $\quad X\sum_{k=1}^{17} 1.04^k = 40{,}000\left[1+v+v^2+v^3\right]$

D $\quad X\sum_{k=1}^{18} 1.04^k = 40{,}000\left[1+v+v^2+v^3\right]$

E $\quad X\sum_{k=0}^{18} v^k = 40{,}000\left[v^{19}+v^{20}+v^{21}+v^{22}\right]$

Question 6.20

Justin turns 40 today and wishes to have retirement income of $4,000 at the beginning of each month, starting on his 65th birthday.

Starting today, he makes monthly contributions of X to a fund for 25 years. The fund earns a nominal interest rate of 5% compounded monthly.

Each $1,000 in the fund will provide $7.25 of monthly income at the beginning of each month, starting on his 65th birthday and lasting until the end of his life.

Calculate X.

A 922.63 B 926.47 C 930.33 D 938.11 E 941.94

Question 6.21

An annuity-due makes payments of $20 every other year, with the first payment beginning immediately. The annuity-due makes ten payments. The nominal annual interest rate compounded semiannually is 6%.

Calculate the present value of the annuity-due.

A 98.18 B 110.50 C 119.01 D 124.37 E 164.97

Question 6.22

An annuity-due makes payments of $1 per year for 5 years, $2 per year for the next 5 years, and $3 per year for the last five years. The annual effective interest rate is 7%.

Calculate the accumulated value of the 15 payments at the end of 15 years.

A 16.20 B 17.33 C 37.94 D 44.70 E 47.82

Question 6.23

Fred will deposit $5,000 on June 1, 2025 and $29,687.69 on June 1, 2040 into an account that accumulates at an annual effective interest rate of 9%.

Ethel will deposit $2,000 at the beginning of each year for n years into an account that accumulates at an annual effective rate of interest of 7%. Ethel's first deposit will be made on June 1, 2025.

On June 1, 2050, the accumulated value in Fred's account will be equal to the accumulated value in Ethel's account.

You are given that n is less than 25. Calculate n.

A 8 B 9 C 13 D 17 E 19

Question 6.24

Ella made a deposit of 500 into a fund at the beginning of each year for 15 years.

At the end of 15 years, she began making semiannual withdrawals of 2,000 at the beginning of each six months, with a smaller final withdrawal to exhaust the fund.

The fund earned an annual effective interest rate of 12%.

Calculate the amount of the final withdrawal.

A 197.80 B 209.33 C 221.54 D 234.45 E 248.12

Question 6.25

College tuition is 7,000 for the current school year, and it increases by 4% per year. College tuition is due in full at the beginning of each school year.

Parents set up a college fund that earns interest at an annual effective rate of 8%. They make deposits of 700 into the fund at the beginning of each school year for 18 years. The first deposit is made at the beginning of the current school year.

Immediately after making the 18th deposit, the parents make the first tuition payment.

The amount of money needed, in addition to the balance of the fund, to make the second tuition payment at the beginning of the 19th school year is X.

Calculate X.

A 496.37 B 594.46 C 1,600.85 D 1,670.53 E 1,750.73

Question 6.26

A perpetuity makes payments of X at the beginning of each year. Amy, Beth, and Candy share the perpetuity such that Amy receives the payments of X for the first n years, and Beth receives the payments of X for the next m years, after which Candy receives all of the remaining payments of X.

Which of the following represents the difference between the present value of Amy's payments and Beth's payments using a constant rate of interest?

A $X\left[a_{\overline{n|}} - v^n a_{\overline{m|}}\right]$

B $X\left[\ddot{a}_{\overline{n|}} - v^{n-1} a_{\overline{m|}}\right]$

C $X\left[a_{\overline{n|}} - v^{n+1} a_{\overline{m|}}\right]$

D $X\left[a_{\overline{n|}} - v^{n-1} \ddot{a}_{\overline{m|}}\right]$

E $X\left[v\ddot{a}_{\overline{n|}} - v^n a_{\overline{m|}}\right]$

Question 6.27

A perpetuity-immediate pays X per year. Aaron receives the first n payments, Brad receives the next $2n$ payments, and Charlie receives the remaining payments.

Aaron's share of the present value of the original perpetuity is 30%, and Charlie's share is K.

Calculate K.

A 0.150 B 0.270 C 0.322 D 0.343 E 0.490

Question 6.28

A deferred perpetuity-due begins making annual payments of 500 per year in 7 years.

The annual interest rate compounded monthly is 7%.

Calculate the present value of the deferred perpetuity-due.

A 3,635 B 4,277 C 4,550 D 4,760 E 4,879

Question 6.29

The following three sets of cash flows have the same present value:

- 10,000 payable in 1 year
- 1,200 at the end of each year for 12 years, with the first payment to be made in one year
- X per year, with the first payment to be made in 20 years and the final payment to be made in 30 years

The force of interest is constant.

Calculate X.

A 281 B 4,971 C 5,299 D 5,341 E 5,693

Question 6.30

Leonard inherited a perpetuity-due with annual payments of 20,000. He immediately exchanged the perpetuity for a 30-year annuity-due having the same present value. The annuity makes annual payments of X.

All of the present values are based on an annual effective interest rate of 9% for the first 8 years and 11% thereafter.

Calculate X.

A 19,232 B 19,820 C 19,944 D 20,251 E 20,963

Chapter 7: Level Annuities Payable More than Once per Time Unit

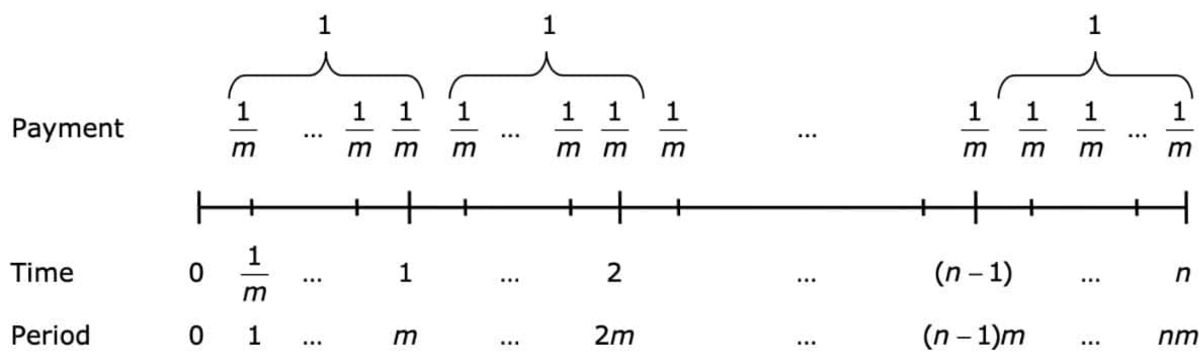
7.01 Annuity-Immediate Payable m^{th}ly

It is common for recurring payments to be expressed as a rate per unit of time, even though the payments can be paid more frequently than that unit of time. For example, an annuity of \$12,000 per year could be paid at a rate of \$1,000 per month. In this example, the unit of time is one year and the **payment period**, which is the period of time between payments, is one month.

Consider an annuity-immediate that pays 1 per time unit, in increments of $1/m$ at the end of each period, where each period is of length $1/m$.

If each period is one month, then m is equal to 12.

As shown in the top row in the figure below, the annuity-immediate pays at a rate of 1 per unit of time:



If the effective annual interest rate is i, then the discount factor for a period of length $1/m$ is:

$$v^{\frac{1}{m}} = \left(\frac{1}{1+i}\right)^{\frac{1}{m}} = \left(\frac{1}{\left(1+\frac{i^{(m)}}{m}\right)^{m}}\right)^{\frac{1}{m}} = \frac{1}{1+\frac{i^{(m)}}{m}}$$

The annuity-immediate pays for nm periods of length $1/m$, and the effective interest rate for a period of length $1/m$ is $i^{(m)}/m$, so the present value of the annuity-immediate can be found by setting the unit of time equal to the length of a period:

$$PV_0 = \frac{1}{m}a_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{1}{m}v^{\frac{1}{m}} + \frac{1}{m}v^{\frac{2}{m}} + \cdots + \frac{1}{m}v^{\frac{nm-1}{m}} + \frac{1}{m}v^{\frac{nm}{m}} = \frac{1}{m}\left[\frac{v^{\frac{1}{m}} - v^{\frac{nm+1}{m}}}{1-v^{\frac{1}{m}}}\right]$$

$$= \frac{1}{m}\left(\frac{1+\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}\right)\left[\frac{v^{\frac{1}{m}} - v^{\frac{nm+1}{m}}}{1-v^{\frac{1}{m}}}\right] = \frac{1}{m}\left[\frac{1-v^{\frac{nm}{m}}}{\frac{i^{(m)}}{m}}\right] = \frac{1}{m}\left[\frac{1-\left(v^{\frac{1}{m}}\right)^{nm}}{\frac{i^{(m)}}{m}}\right]$$

Since $\left(v^{\frac{1}{m}}\right)^{nm} = v^n$, the final expression above can be written as:

$$\frac{1}{m}\left[\frac{1-v^n}{\frac{i^{(m)}}{m}}\right]$$

If we simplify, we have:

$$\frac{1}{m}\left[\frac{1-v^n}{\frac{i^{(m)}}{m}}\right] = \frac{1-v^n}{i^{(m)}}$$

This annuity is described as an annuity-immediate payable m^{th}ly, and the notation for its present value is:

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}}$$

Including the effective interest rate per time unit, as measured by the number of units inside the right-angle bracket, is optional, but the expression above can also be written as:

$$a_{\overline{n}|i}^{(m)} = \frac{1-v^n}{i^{(m)}}$$

Including the effective interest rate per time unit is also optional for the following annuity notation introduced in this chapter: $\ddot{a}_{\overline{n}|i}^{(m)}$, $s_{\overline{n}|i}^{(m)}$, and $\ddot{s}_{\overline{n}|i}^{(m)}$.

The notation for the accumulated value of the annuity-immediate at time n is $s_{\overline{n}|}^{(m)}$, and it is equal to the accumulated value of the present value:

$$AV_n = s_{\overline{n}|}^{(m)} = (1+i)^n \times \frac{1-v^n}{i^{(m)}} = \frac{(1+i)^n - 1}{i^{(m)}}$$

Alternatively, the accumulated value can be found by noting that there are nm periods, each of which have an effective interest rate of $\frac{i^{(m)}}{m}$:

$$\frac{1}{m} s_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{1}{m} \frac{\left(1+\frac{i^{(m)}}{m}\right)^{nm} - 1}{\frac{i^{(m)}}{m}} = \frac{1}{m} \times \frac{(1+i)^n - 1}{\frac{i^{(m)}}{m}}$$

Annuity-Immediate Payable m^{th}ly

7.01

The present value at time 0 and the accumulated value at time n of an annuity-immediate that makes payments of 1 per time unit, payable m^{th}ly, for n units of time are:

$$a_{\overline{n}|}^{(m)} = \frac{1}{m} a_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{1-v^n}{i^{(m)}}$$

$$s_{\overline{n}|}^{(m)} = \frac{1}{m} s_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{(1+i)^n - 1}{i^{(m)}}$$

Example 7.01

The nominal annual interest rate compounded monthly is 12%. Find the present value of an annuity-immediate that makes monthly payments at a rate of $36 per year for 10 years.

Solution The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

There are two ways to answer this question.

1. The annual rate of payment is \$36 per year, so we have:

$$36 \times a_{\overline{10}|}^{(12)} = 36 \times \frac{1 - v^{10}}{i^{(12)}} = 36 \times \frac{1 - (1.01)^{-12\times10}}{0.12} = 36 \times 5.8083$$
$$= \mathbf{209.10}$$

where:

$$v = \left(\frac{1}{1.01}\right)^{12}$$

2. Alternatively, we can focus on the monthly payment of \$3:

$$36 \times \frac{1}{12} a_{\overline{10\times12}|\frac{i^{(12)}}{12}} = 3 \times a_{\overline{120}|0.01} = 3 \times \left[\frac{1 - v^{120}}{0.01}\right] = 3 \times \left[\frac{1 - (1.01)^{-120}}{0.01}\right]$$
$$= 3 \times 69.7005 = \mathbf{209.10}$$

where:

$$v = \frac{1}{1.01}$$

This second method is convenient for use with the BA II Plus:

120 [*N*] 1 [*I/Y*] 3 [*PMT*] [CPT] [*PV*]

PV = –209.10 Answer = **209.10**

The present value of a perpetuity-immediate of 1 per time unit, payable m^{th}ly, is found by taking the limit as n goes to infinity:

$$a_{\overline{\infty}|}^{(m)} = \lim_{n\to\infty} a_{\overline{n}|}^{(m)} = \lim_{n\to\infty} \frac{1 - v^n}{i^{(m)}} = \frac{1}{i^{(m)}}$$

Example 7.02 The nominal annual interest rate compounded monthly is 12%. Find the present value of a perpetuity-immediate that makes monthly payments at a rate of \$36 per year forever.

Solution The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

There are two ways to answer this question.

1. The annual rate of payment is \$36 per year, so we have:

$$36 \times a_{\overline{\infty}|}^{(12)} = 36 \times \frac{1}{i^{(12)}} = 36 \times \frac{1}{0.12} = \mathbf{300}$$

2. Alternatively, we can focus on the monthly payment of \$3:

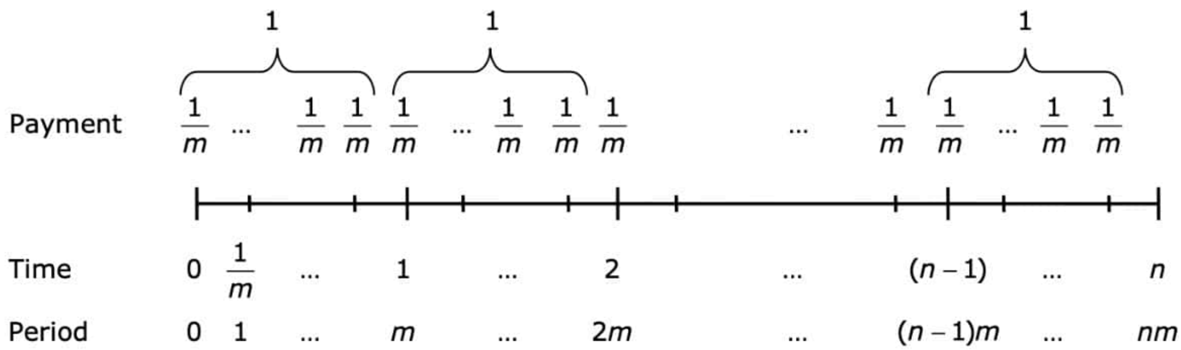
$$3 \times a_{\overline{\infty}|0.01} = 3 \times \frac{1}{0.01} = \mathbf{300}$$

7.02 Annuity-Due Payable m^{th}ly

Consider an annuity-due that pays 1 per time unit, in increments of $1/m$ at the beginning of each period, where each period is of length $1/m$.

If each period is one month, then m is equal to 12.

As shown in the top row in the figure below, the annuity-due pays at a rate of 1 per unit of time:



The discount factor for a period of length $1/m$ is:

$$v^{\frac{1}{m}} = \left(\frac{1}{1+i}\right)^{\frac{1}{m}} = \left(\frac{1}{\left(1+\frac{i^{(m)}}{m}\right)^{m}}\right)^{\frac{1}{m}} = \frac{1}{1+\frac{i^{(m)}}{m}} = 1-\frac{d^{(m)}}{m}$$

The annuity-due makes payments for nm periods of length $1/m$, and the effective interest rate for a period of length $1/m$ is $i^{(m)}/m$, so the present value of the annuity-due can be found by setting the unit of time equal to the length of a period:

$$PV_0 = \frac{1}{m}\ddot{a}_{\overline{nm|}\frac{i^{(m)}}{m}} = \frac{1}{m}+\frac{1}{m}v^{\frac{1}{m}}+\frac{1}{m}v^{\frac{2}{m}}+\cdots+\frac{1}{m}v^{\frac{nm-1}{m}} = \frac{1}{m}\left[\frac{1-v^n}{1-v^{\frac{1}{m}}}\right] = \frac{1}{m}\left[\frac{1-v^n}{\frac{d^{(m)}}{m}}\right]$$

$$= \frac{1-v^n}{d^{(m)}}$$

This annuity is described as an annuity-due payable m^{th}ly, and the notation for its present value is:

$$\ddot{a}^{(m)}_{\overline{n|}} = \frac{1-v^n}{d^{(m)}}$$

The notation for the accumulated value of the annuity-due at time n is $\ddot{s}^{(m)}_{\overline{n|}}$, and it is equal to the accumulated value of the present value:

$$AV_n = \ddot{s}^{(m)}_{\overline{n|}} = (1+i)^n \times \frac{1-v^n}{d^{(m)}} = \frac{(1+i)^n-1}{d^{(m)}}$$

Alternatively, the accumulated value can be found by noting that there are nm periods, each of which have an effective interest rate of $i^{(m)}/m$:

$$\frac{1}{m}\ddot{s}_{\overline{nm|}\frac{i^{(m)}}{m}} = \frac{1}{m}\frac{\left(1+\frac{i^{(m)}}{m}\right)^{nm}-1}{\frac{d^{(m)}}{m}} = \frac{1}{m}\times\frac{(1+i)^n-1}{\frac{d^{(m)}}{m}}$$

7.02

Annuity-Due Payable m^{th}ly

The present value at time 0 and the accumulated value at time n of an annuity-due that makes payments of 1 per time unit, payable m^{th}ly, for n units of time are:

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1}{m}\ddot{a}_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{1-v^n}{d^{(m)}} \qquad \ddot{s}_{\overline{n}|}^{(m)} = \frac{1}{m}\ddot{s}_{\overline{nm}|\frac{i^{(m)}}{m}} = \frac{(1+i)^n-1}{d^{(m)}}$$

Since each of the payments from an annuity-due payable m^{th}ly is made one period earlier than the payments from an otherwise equivalent annuity-immediate payable m^{th}ly, we have:

$$\ddot{a}_{\overline{n}|}^{(m)} = a_{\overline{n}|}^{(m)} \times \left(1+\frac{i^{(m)}}{m}\right) \qquad \ddot{s}_{\overline{n}|}^{(m)} = s_{\overline{n}|}^{(m)} \times \left(1+\frac{i^{(m)}}{m}\right)$$

Example 7.03

The nominal annual interest rate compounded monthly is 12%. Find the accumulated value at the end of 10 years of an annuity-due that makes monthly payments at a rate of \$36 per year for 10 years.

Solution

The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

The annual discount rate compounded monthly is:

$$d^{(12)} = \left[1-\left(1+\frac{i^{(12)}}{12}\right)^{-1}\right]\times 12 = \left[1-(1.01)^{-1}\right]\times 12 = 0.1188$$

We show two ways to answer this question.

1. The annual rate of payment is \$36 per year, so we have:

$$36\times\ddot{s}_{\overline{10}|}^{(12)} = 36\times\frac{(1+i)^n-1}{d^{(p)}} = 36\times\frac{(1.01)^{12\times 10}-1}{0.1188} = 36\times 19.3616$$
$$= \mathbf{697.02}$$

where:

$$i = 1.01^{12}-1$$

2. Alternatively, we can focus on the monthly payment of \$3:

$$36\times\frac{1}{12}\ddot{s}_{\overline{10\times 12}|\frac{i^{(12)}}{12}} = 3\times\ddot{s}_{\overline{120}|0.01} = 3\times\left[\frac{(1.01)^{120}-1}{\frac{0.01}{1.01}}\right] = 3\times 232.3391$$
$$= \mathbf{697.02}$$

where:

$$i = 0.01$$

This second method is convenient for use with the BA II Plus. The method below allows the calculator to be left in the END mode:

120 [*N*] 1 [*I/Y*] 3 [*PMT*] [CPT] [*FV*]

FV = –690.12

[×] 1.01 [=] –697.02

Answer = **697.02**

The present value of a perpetuity-due of 1 per time unit, payable m^{th}ly, is found by taking the limit as n goes to infinity:

$$\ddot{a}_{\overline{\infty}|}^{(m)} = \lim_{n\to\infty} \ddot{a}_{\overline{n}|}^{(m)} = \lim_{n\to\infty} \frac{1-v^n}{d^{(m)}} = \frac{1}{d^{(m)}}$$

The present value of the perpetuity-due can also be written as the present value of a perpetuity-immediate plus an additional payment made now:

$$\ddot{a}_{\overline{\infty}|}^{(m)} = a_{\overline{\infty}|}^{(m)} + \frac{1}{m}$$

Example 7.04 The nominal annual interest rate compounded monthly is 12%. Find the present value of a perpetuity-due that makes monthly payments at a rate of \$36 per year forever.

Solution The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.12}{12} = 0.01$$

The annual discount rate compounded monthly is:

$$d^{(12)} = \left[1-\left(1+\frac{i^{(12)}}{12}\right)^{-1}\right] \times 12 = \left[1-(1.01)^{-1}\right] \times 12 = 0.1188$$

We show three ways to answer this question.

1. The annual rate of payment is \$36 per year, so we have:

$$36 \times \ddot{a}_{\overline{\infty}|}^{(12)} = 36 \times \frac{1}{d^{(12)}} = 36 \times \frac{1}{0.1188} = \mathbf{303}$$

2. Alternatively, we can focus on the monthly payment of \$3:

$$3 \times \ddot{a}_{\overline{\infty}|0.01} = 3 \times \frac{1}{\frac{d^{(12)}}{12}} = 3 \times \frac{1}{\frac{0.01}{1.01}} = \mathbf{303}$$

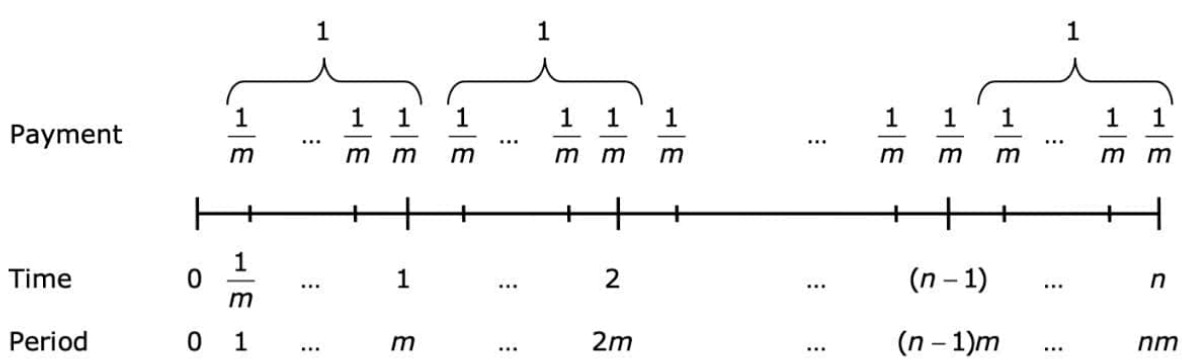
3. An easy way to answer the question is to recognize that the perpetuity-due is a perpetuity-immediate plus a payment of \$3 now:

$$3 \times \ddot{a}_{\overline{\infty}|}^{(12)} = 3 + 3 \times a_{\overline{\infty}|}^{(12)} = 3 + \frac{3}{0.01} = \mathbf{303}$$

7.03 Level Annuities Payable Continuously

In Section 7.01, we found the present value of an annuity-immediate that pays 1 per time unit, in increments of $1/m$ at the end of each period:

$$a_{\overline{n}|}^{(m)} = \frac{1-v^n}{i^{(m)}}$$



As m becomes larger, the payments are made more frequently, but the rate of payment remains 1 per time unit:

$$\left[\text{Payment per period}\right] \times \left[\begin{array}{c}\text{Number of periods}\\ \text{per time unit}\end{array}\right] = \text{Payment rate per time unit}$$

$$\frac{1}{m} \times m = 1$$

As m approaches infinity, the rate of payment becomes continuous.

If you've seen a post-apocalyptic movie in which water or gasoline has become so precious that it is used as currency, then you can picture continuous payments!

In Section 4.03, we saw that if the force of interest is held constant, then as the compounding frequency approaches infinity, the nominal interest rate approaches the continuously compounded interest rate:

$$\lim_{m \to \infty} i^{(m)} = r$$

The notation for the present value of an annuity that continuously pays one per time unit is $\bar{a}_{\overline{n}|}$. This present value can be found by taking the limit of the present value of an annuity that pays 1 per time unit, payable m^{th}ly, as m increases to infinity:

$$\bar{a}_{\overline{n}|} = \lim_{m \to \infty} a_{\overline{n}|}^{(m)} = \lim_{m \to \infty} \frac{1 - v^n}{i^{(m)}} = \frac{1 - v^n}{r}$$

The accumulated value of the annuity at time n is denoted by $\bar{s}_{\overline{n}|}$, and it can be found by accumulating the present value for n units of time:

$$\bar{s}_{\overline{n}|} = (1+i)^n \bar{a}_{\overline{n}|} = (1+i)^n \frac{1 - v^n}{r} = \frac{(1+i)^n - 1}{r}$$

7.03

Continuously Payable Annuity

The present value at time 0 and the accumulated value at time n of a continuously payable annuity that makes payments of 1 per time unit, for n units of time are:

$$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{r} \qquad \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{r}$$

where:

$r =$ Force of interest

Example 7.05

The nominal annual interest rate compounded monthly is 12%. Find the present value of an annuity that makes continuous payments at a rate of \$36 per year for 10 years.

Solution

The continuously compounded interest rate is:

$$r = \ln\left(1 + \frac{i^{(12)}}{12}\right)^{12} = \ln\left(1 + \frac{0.12}{12}\right)^{12} = 0.1194$$

The annual rate of payment is \$36 per year, so we have:

$$36 \times \bar{a}_{\overline{10}|} = 36 \times \frac{1 - v^{10}}{r} = 36 \times \frac{1 - (1.01)^{-12 \times 10}}{0.1194} = 36 \times 5.874 = \mathbf{210.15}$$

The present value of a continuously payable perpetuity of 1 per time unit is found by taking the limit as n goes to infinity:

$$\bar{a}_{\overline{\infty}|} = \lim_{n \to \infty} \bar{a}_{\overline{n}|} = \lim_{n \to \infty} \frac{1 - v^n}{r} = \frac{1}{r}$$

7.04 Questions

Question 7.01

Paul will receive payments of 50 every three months for 10 years. The first payment will be made 3 months from today. The annual effective interest rate is 8%.

Calculate the present value of the annuity.

A 1,032.89 B 1,272.23 C 1,367.77 D 1,381.63 E 1,992.06

Question 7.02

A perpetuity makes quarterly payments of 45. The next payment will be made in two months. The quarterly effective interest rate is 3%.

Calculate the present value of the perpetuity.

A 1,485.29 B 1,500.00 C 1,511.13 D 1,514.85 E 1,529.85

Question 7.03

An annuity-immediate pays 600 quarterly for 12 years.

Jeremy purchases the annuity for $15,000.

Calculate the nominal interest rate convertible monthly that Jeremy earns.

A 12.11% B 12.23% C 12.80% D 13.23% E 13.90%

Question 7.04

David has an annuity-immediate that makes annual payments of 1,000 for 10 years.

Rebecca has an annuity-immediate that makes level monthly payments for 10 years.

At an annual effective interest rate of 8%, the present value of David's annuity is equal to the present value of Rebecca's annuity.

David invests his annuity payments at an annual effective rate of 9% per year.

Rebecca invests her annuity payments at an annual effective rate of 10% per year.

Calculate the difference between the accumulated value of Rebecca's payments and the accumulated value of David's payments at the end of 10 years.

A 65 B 137 C 706 D 881 E 1,588

Question 7.05

Doug buys a recreational vehicle and finances it with a loan of 30,000. The annual interest rate on the loan is 9% compounded monthly.

Doug makes 22 monthly payments of 700. The first payment is made one month after purchasing the recreational vehicle. Immediately after the 22^{nd} payment, he refinances the loan with a new loan that calls for him to make 30 monthly payments. The first of these payments is to be made one month later. The refinanced loan has a nominal interest rate of 6% compounded monthly.

Calculate the amount of the new monthly payment.

A 578.82 B 623.30 C 672.24 D 697.83 E 898.98

Question 7.06

John buys a house and finances it with a loan of 294,584.81. The annual interest rate on the loan is 7.2% compounded monthly.

John makes monthly payments of 2,000 for five years, and the first payment is made one month after purchasing the house. Immediately after the 60th payment, he refinances the loan with a new loan that calls for him to make *n* monthly payments, with the first payment to be made one month later. The refinanced loan has a nominal interest rate of 3.6% compounded monthly. The monthly payment amount does not change.

Calculate the number of months needed to pay off the refinanced loan.

A 116 B 120 C 150 D 175 E 180

Question 7.07

The present value of a perpetuity paying 15 at the end of every 4-year period, with the first payment made at the end of year 4, is 37.50.

Using the same annual effective interest rate, find the present value of a perpetuity paying 1 at the end of each 4-month period, with the first payment made at the end of 4 months.

A 30.98 B 35.17 C 36.17 D 41.47 E 47.05

Question 7.08

Laura's bank account pays an annual nominal interest rate of 5.4% convertible monthly.

On January 1 of year *y*, Laura's account balance was *X*. Laura then deposited 1,000 at the end of every quarter. On August 1, of year $(y + 5)$, the account balance was $1.7X$.

Which of the following equations is an equation of value that can be used to solve for *X*?

A $X + \sum_{k=1}^{22} \frac{1{,}000}{1.0045^{3k}} = \frac{1.7X}{1.0045^{68}}$

B $X + \sum_{k=1}^{22} \frac{1{,}000}{1.0045^{3k}} = \frac{1.7X}{1.0045^{67}}$

C $X + \sum_{k=1}^{23} \frac{1{,}000}{1.0045^{3k}} = \frac{1.7X}{1.0045^{67}}$

D $X + \sum_{k=1}^{23} \frac{1{,}000}{1.0045^{3k}} = \frac{1.7X}{1.0045^{68}}$

E $X + \sum_{k=0}^{22} \frac{1{,}000}{1.0045^{3k}} = \frac{1.7X}{1.0045^{68}}$

Question 7.09

Laura's bank account pays an annual nominal interest rate of 5.4% convertible monthly.

On January 1 of year *y*, Laura's account balance was *X*. Laura then deposited 1,000 at the end of every quarter. On August 1, of year $(y + 5)$, the account balance was $1.7X$.

Calculate *X*.

A 36,571 B 36,735 C 54,428 D 71,631 E 73,201

Question 7.10

Mildred will receive payments of 50 every three months for 10 years. The first payment is made today. The annual effective interest rate is 8%.

Calculate the present value of the annuity.

A 1,059.73 B 1,358.47 C 1,381.63 D 1,395.13 E 1,408.47

Question 7.11

A perpetuity-immediate will make quarterly payments of 45. The next payment will be made in three months. The quarterly effective interest rate is 3%.

A perpetuity-due will make monthly payments of *X*. The next payment will be made immediately.

The present value of the perpetuity-immediate is equal to the present value of the perpetuity-due.

Calculate *X*.

A 3.70 B 11.04 C 14.71 D 14.85 E 43.69

Question 7.12

A marina offers two ways to pay for a boat:

1. Regular option: Pay full price for the boat now.
2. Financing option: Divide the price by 11 and pay that amount at the beginning of each month for 12 months.

Determine the annual effective interest rate paid by the customers that choose the financing option.

A 15.8% B 16.4% C 17.7% D 19.5% E 21.3%

Question 7.13

Susan makes payments into a fund at the beginning of each month for 9 years. For the first 3 years, her monthly payments are 100. For the next 3 years, her monthly payments are 200. For the final 3 years, her payments are 300. At the end of 11 years, the value of the fund is 30,000.

The annual effective interest rate is *i*, and the monthly effective interest rate is *j*.

Based on the equation of value for time 11, which of the following equations is valid?

A $s_{\overline{36}|j}(1+i)^2\left[(1+i)^6+2(1+i)^3+3\right]=300$

B $\ddot{s}_{\overline{36}|i}(1+i)^2\left[(1+i)^6+2(1+i)^3+3\right]=300$

C $s_{\overline{36}|i}(1+j)^2\left[(1+i)^6+2(1+i)^3+3\right]=300$

D $\ddot{s}_{\overline{36}|j}(1+i)^2\left[(1+i)^6+2(1+i)^3+3\right]=300$

E $\ddot{s}_{\overline{36}|j}(1+i)^2\left[(1+j)^6+2(1+j)^3+3\right]=300$

Question 7.14

Harry has an annuity-due that makes annual payments of 1,000 for 10 years.

Minnie has an annuity-due that makes level monthly payments for 10 years.

At an annual effective interest rate of 8%, the present value of Harry's annuity is equal to the present value of Minnie's annuity.

Harry invests his annuity payments at an annual effective rate of 9% per year.

Minnie invests her annuity payments at an annual effective rate of 10% per year.

Calculate the difference between the accumulated value of Minnie's payments and the accumulated value of Harry's payments at the end of 10 years.

A 144.55 B 826.33 C 881.20 D 1,266.00 E 1,741.13

Question 7.15

A woman turns 37 today, and she wants to have retirement income of 2,000 at the beginning of each month, starting when she turns 70. Starting today, she makes monthly contributions of X to a fund for 33 years. The fund earns a nominal annual interest rate of 7% compounded monthly.

On her 70th birthday, each 1,000 of the fund will provide 9.45 of income at the beginning of each month, starting immediately and continuing as long as she lives.

Calculate X.

A 136.27 B 137.07 C 142.93 D 143.75 E 144.07

Question 7.16

Deposits of 250 are made into an account at the beginning of each 5-year period for 40 years. The account earns interest at an annual effective interest rate of i.

The accumulated amount at the end of 40 years is X, which is 4 times the accumulated amount in the account at the end of 20 years.

Calculate X.

A 6,327.63 B 8,327.63 C 22,418.72 D 35,418.72 E 37,418.75

Question 7.17

The present value of a perpetuity paying 1 every three years, with the first payment due immediately, is 6.74 at an annual effective rate of i.

Another perpetuity pays X every 4 years, with the first payment due at the beginning of year 2. This perpetuity has the same present value at an annual effective rate of $(i + 0.01)$.

Calculate X.

A 1.0 B 1.6 C 1.7 D 1.9 E 2.3

Question 7.18

Five deposits of X are made into a fund at two-year intervals with the first deposit at the beginning of the first year.

The fund earns interest at an annual effective rate of 5% during the first six years, and it earns interest at an annual effective rate of 7% thereafter.

Calculate the equivalent level annual effective yield earned over the 10-year interval.

A 5.76% B 5.90% C 6.14% D 7.50% E 8.94%

Question 7.19

Teresa pays for her son's college education by depositing X into a fund at the beginning of each month for 20 years. The fund earns an annual effective interest rate of 7%.

Teresa withdraws 35,000 from the fund at the beginning of the 18th, 19th, 20th, and 21st years. After the final withdrawal, the balance of the fund is zero.

Calculate X.

A 265.93 B 284.54 C 286.15 D 304.46 E 325.77

Question 7.20

A perpetuity makes continuous payments at a rate of \$50 per year. The annual continuously compounded interest rate is 7%.

Calculate the present value of the perpetuity.

A 665.72 B 689.58 C 695.72 D 714.29 E 739.00

Question 7.21

A 10-year annuity makes continuous payments at a rate of \$50 per year. The annual continuously compounded interest rate is 7%.

Calculate the accumulated value of the annuity at the end of 10 years.

A 351.18 B 359.58 C 666.93 D 690.82 E 724.11

Question 7.22

The force of interest is denoted by δ.

You are given:

$$\bar{a}_{\overline{10|}} = 8.1277 \quad \text{and} \quad \frac{d\bar{a}_{\overline{10|}}}{d\delta} = -37.735$$

Calculate δ.

A 4.0% B 4.1% C 4.2% D 4.3% E 4.4%

Question 7.23

You are given:

$$\ddot{a}_{\overline{n+2|}} = 13.0685 \quad \text{and} \quad \ddot{s}_{\overline{n|}} = 17.3958$$

Calculate $\bar{a}_{\overline{1|}}$.

A 0.986 B 0.988 C 0.990 D 0.993 E 0.995

Chapter 8: Arithmetic Progression Annuities

An arithmetic progression annuity is an annuity with payments that increase by a constant amount at regular intervals.

8.01 Increasing Annuity

Consider an annuity-immediate whose payments increase by 1 per unit of time. The annuity pays 1 at the end of the first unit of time. At the end of the second unit of time, it pays 2. At the end of the third time unit, it pays 3. This continues until time n, when the annuity makes its final payment of n.

Payment		1	2	3	...	$n-1$	n
Time	0	1	2	3	...	$n-1$	n

We use $(Ia)_{\overline{n|}}$ to denote the present value of this increasing annuity-immediate:

$$(Ia)_{\overline{n|}} = v + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1} + nv^n$$

To obtain a convenient expression for $(Ia)_{\overline{n|}}$, we multiply the expression above by $(1+i)$ and then subtract the original expression from both sides:

$$(1+i)(Ia)_{\overline{n|}} = 1 + 2v + 3v^2 + \cdots + (n-1)v^{n-2} + nv^{n-1}$$

$$(Ia)_{\overline{n|}} = \quad v + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1} + nv^n$$

$$(1+i)(Ia)_{\overline{n|}} - (Ia)_{\overline{n|}} = 1 + v + v^2 + \cdots + v^{n-1} - nv^n$$

$$(Ia)_{\overline{n|}} \times i = \ddot{a}_{\overline{n|}} - nv^n$$

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i}$$

The accumulated value of the increasing annuity-immediate at time n is denoted by $(Is)_{\overline{n|}}$ and it is found below by accumulating the present value for n years:

$$(Is)_{\overline{n|}} = (1+i)^n (Ia)_{\overline{n|}} = (1+i)^n \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} = \frac{\ddot{s}_{\overline{n|}} - n}{i}$$

8.01

Increasing Annuity-Immediate

The present value at time 0 and the accumulated value at time n of an increasing annuity-immediate that pays 1 at the time 1, 2 at time 2, and so on until the final payment of n at time n are:

$$(Ia)_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} \qquad\qquad (Is)_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{i}$$

Example 8.01

An annuity-immediate pays \$10 at the end of 1 year, \$20 at the end of 2 years, \$30 at the end of 3 years, and so on, until a final payment of \$120 is made at the end of 12 years. The annual effective interest rate is 9% per year. Calculate the present value of the annuity.

Solution

The present value is:

$$10v+20v^2+\cdots+120v^{12}=10(v+2v^2+\cdots+12v^{12})=10(Ia)_{\overline{12|}}$$
$$=10\times\frac{\ddot{a}_{\overline{12|}}-12v^{12}}{0.09}$$
$$=10\times\frac{\frac{1-1.09^{-12}}{0.09}\times1.09-12\times1.09^{-12}}{0.09}$$
$$=10\times\frac{7.8052-12\times1.09^{-12}}{0.09}=10\times39.3197$$
$$=\mathbf{393.20}$$

The present value in the Key Concept above can be broken down into two parts:

$$(Ia)_{\overline{n|}}=\frac{\ddot{a}_{\overline{n|}}-nv^n}{i}=\frac{1}{i}\ddot{a}_{\overline{n|}}-\frac{n}{i}v^n$$

Using the BA II Plus, we begin by setting the calculator to the BGN mode:

[2nd] [BGN]   [2nd] [SET]   [2nd] [QUIT]

12 [*N*]   9 [*I/Y*]   0.09 [1/*x*] [*PMT*]   12 [÷] 0.09 [=] [+/-] [*FV*]   [CPT] [*PV*]

$PV = -39.31971$

[+/-] [×] 10   Answer = **393.20**

In the example below, the payments increase monthly instead of annually.

Example 8.02

An annuity-immediate pays \$100 at the end of 1 month, \$110 at the end of 2 months, \$120 at the end of 3 months, and so on for 15 years.

The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution

Since the payments are made monthly and increase monthly, we use one month as our unit of time. The monthly effective interest rate is:

$$\frac{0.06}{12}=0.005$$

The present value of the annuity is equal to the present value of two annuities:

- an annuity that pays \$90 per month, beginning at the end of 1 month and continuing for 180 months.
- an increasing annuity that pays \$10 at the end of 1 month, \$20 at the end of 2 months, \$30 at the end of 3 months, and so on for 180 months.

The present value is:

$$PV_0=90a_{\overline{180|}0.005}+10(Ia)_{\overline{180|}0.005}$$
$$=90\times\frac{1-1.005^{-180}}{0.005}+10\times\frac{\ddot{a}_{\overline{180|}}-180\times1.005^{-180}}{0.005}$$
$$=90\times118.5035+10\times\frac{118.5035\times1.005-73.3468}{0.005}$$
$$=10{,}665.3163+10\times9{,}149.8391=\mathbf{102{,}163.71}$$

If an increasing annuity makes each of its payments at the beginning of each unit of time, then it is called an increasing annuity-due. Consider an annuity that pays 1 at the beginning of the first unit of time, 2 at the beginning of the second unit of time, and so on, until it makes its final payment of n at the beginning of the n^{th} unit of time.

Payment	1	2	3	4	...	n	
Time	0	1	2	3	...	$n-1$	n

We use $(I\ddot{a})_{\overline{n|}}$ to denote the present value of this increasing annuity-due:

$$(I\ddot{a})_{\overline{n|}} = 1 + 2v + 3v^2 + \cdots + (n-1)v^{n-2} + nv^{n-1}$$

We can accumulate each of the payments to the end of each time unit to obtain a stream of payments that has the same present value. Since $\ddot{s}_{\overline{1|}} = 1 + i$, the following increasing annuity-immediate has the same present value as the annuity-due shown above:

Payment		$1\,\ddot{s}_{\overline{1|}}$	$2\,\ddot{s}_{\overline{1|}}$	$3\,\ddot{s}_{\overline{1|}}$	...	$(n-1)\,\ddot{s}_{\overline{1|}}$	$n\,\ddot{s}_{\overline{1|}}$
Time	0	1	2	3	...	$n-1$	n

We can now write:

$$\begin{aligned}
(I\ddot{a})_{\overline{n|}} &= 1 + 2v + 3v^2 + \cdots + (n-1)v^{n-2} + nv^{n-1} \\
&= v\left[1\ddot{s}_{\overline{1|}} + 2v\ddot{s}_{\overline{1|}} + 3v^2\ddot{s}_{\overline{1|}} + \cdots + (n-1)v^{n-2}\ddot{s}_{\overline{1|}} + nv^{n-1}\ddot{s}_{\overline{1|}}\right] \\
&= \ddot{s}_{\overline{1|}}\left[v + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1} + nv^n\right] \\
&= \ddot{s}_{\overline{1|}}(Ia)_{\overline{n|}} = \ddot{s}_{\overline{1|}} \times \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} = \frac{i}{d} \times \frac{\ddot{a}_{\overline{n|}} - nv^n}{i} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{d}
\end{aligned}$$

The accumulated value of the increasing annuity-due at time n is denoted by $(I\ddot{s})_{\overline{n|}}$ and it is found below by accumulating the present value for n years:

$$(I\ddot{s})_{\overline{n|}} = (1+i)^n (I\ddot{a})_{\overline{n|}} = (1+i)^n \frac{\ddot{a}_{\overline{n|}} - nv^n}{d} = \frac{\ddot{s}_{\overline{n|}} - n}{d}$$

8.02

Increasing Annuity-Due

The present value at time 0 and the accumulated value at time n of an increasing annuity-due that pays 1 at time 0, 2 at time 1, and so on until a final payment of n is made at time $(n-1)$ are:

$$(I\ddot{a})_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{d} \qquad\qquad (I\ddot{s})_{\overline{n|}} = \frac{\ddot{s}_{\overline{n|}} - n}{d}$$

Example 8.03

An annuity-due pays \$10 now, \$20 at the end of 1 year, \$30 at the end of 2 years, and so on, until a final payment of \$120 is made at the end of 11 years. The annual effective interest rate is 9% per year. Calculate the present value of the annuity.

Solution The present value is:

$$10 + 20v + 30v^2 + \cdots + 120v^{11} = 10(1 + 2v^2 + \cdots + 12v^{11}) = 10(I\ddot{a})_{\overline{12|}}$$

$$= 10 \times \frac{\ddot{a}_{\overline{12|}} - 12v^{12}}{d} = 10 \times \frac{\ddot{a}_{\overline{12|}} - 12v^{12}}{\frac{0.09}{1.09}}$$

$$= 10 \times \frac{\frac{1 - 1.09^{-12}}{0.09} \times 1.09 - 12 \times 1.09^{-12}}{0.0826}$$

$$= 10 \times \frac{7.8052 - 12 \times 1.09^{-12}}{0.0826} = 10 \times 42.8585$$

$$= \mathbf{428.58}$$

The present value in the Key Concept above can be broken down into two parts:

$$(I\ddot{a})_{\overline{n|}} = \frac{\ddot{a}_{\overline{n|}} - nv^n}{d} = \frac{1}{d}\ddot{a}_{\overline{n|}} - \frac{n}{d}v^n = \frac{1.09}{0.09}\ddot{a}_{\overline{12|}} - \frac{12 \times 1.09}{0.09}v^{12}$$

Using the BA II Plus, we begin by setting the calculator to the BGN mode:

[2nd] [BGN]   [2nd] [SET]   [2nd] [QUIT]
12 [*N*]   9 [*I/Y*]   1.09 [÷] 0.09 [=] [*PMT*]
12 [÷] 0.09 [×] 1.09 [=] [+/-] [*FV*]   [CPT] [*PV*]
PV = –42.85848
[+/-] [×] 10   Answer = **428.58**

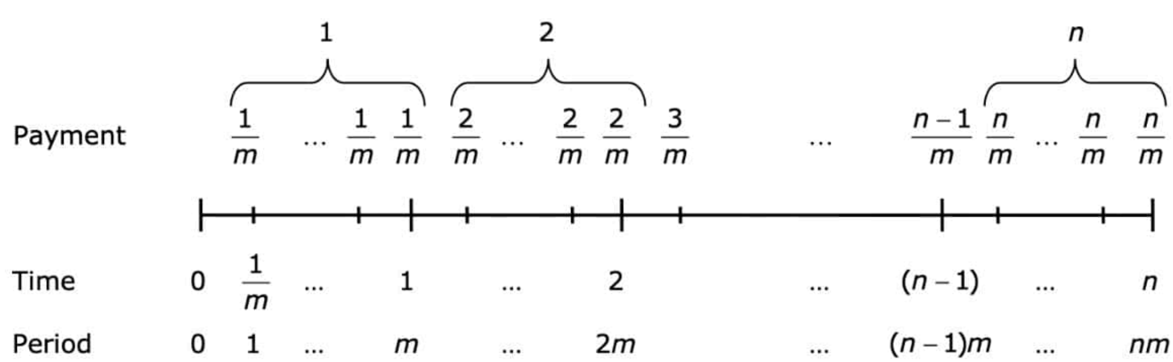
Alternatively, we can treat the increasing annuity as an increasing annuity-immediate, and then multiply by 1.09 at the end. We begin by setting the calculator to the BGN mode:

[2nd] [BGN]   [2nd] [SET]   [2nd] [QUIT]
12 [*N*]   9 [*I/Y*]   1 [÷] 0.09 [=] [*PMT*]   12 [÷] 0.09 [=] [+/-] [*FV*]
[CPT] [*PV*]
PV = –39.31971   [+/-] [×] 10 [×] 1.09 [=]
Answer = **428.58**

We can value an annuity that is payable m^{th}ly and increases at the end of each unit of time. For example, if the unit of time is one year and m = 12, then the annuity payments increase annually and are payable monthly.

Consider an annuity that makes payments m^{th}ly at a rate of 1 per unit of time in the first unit of time, 2 per unit of time in the second unit of time, and so on until payments of n per unit of time are paid in the n^{th} unit of time.

The annuity pays $1/m$ at times $1/m$, $2/m$, …, $(m-1)/m$, and 1. It pays $2/m$ at times $(m+1)/m$, $(m+2)/m$, …, $(2m-1)/m$, and 2. This continues until the n^{th} unit of time, when the annuity pays n/m at times $((n-1)m+1)/m$, $((n-1)m+2)/m$, …, $(nm-1)/m$, and n:



The time 0 present value of this increasing annuity-immediate, payable m^{th}ly is denoted by $(Ia)_{\overline{n}|}^{(m)}$.

We can accumulate the payments to the end of each unit of time to obtain a stream of payments that has the same present value. The following annuity-immediate has the same present value as the annuity-immediate shown above:

Payment		$1\,s_{\overline{1}|}^{(m)}$	$2\,s_{\overline{1}|}^{(m)}$	$3\,s_{\overline{1}|}^{(m)}$	...	$(n-1)s_{\overline{1}|}^{(m)}$	$ns_{\overline{1}|}^{(m)}$
Time	0	1	2	3	...	$n-1$	n

The present value is equal to the present value of a series of annuities, each of which is payable m^{th}ly:

$$(Ia)_{\overline{n}|}^{(m)} = vs_{\overline{1}|}^{(m)} + 2v^2 s_{\overline{1}|}^{(m)} + 3v^3 s_{\overline{1}|}^{(m)} + \cdots + nv^n s_{\overline{1}|}^{(m)}$$

$$= s_{\overline{1}|}^{(m)}\left[v + 2v^2 + 3v^3 + \cdots + nv^n\right] = s_{\overline{1}|}^{(m)}(Ia)_{\overline{n}|} = \frac{(1+i)-1}{i^{(m)}} \times (Ia)_{\overline{n}|}$$

$$= \frac{i}{i^{(m)}} \times (Ia)_{\overline{n}|} = \frac{i}{i^{(m)}} \times \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}}$$

The accumulated value as of time n is denoted by $(Is)_{\overline{n}|}^{(m)}$, and it is found by accumulating the present value for n units of time:

$$(Is)_{\overline{n}|}^{(m)} = (1+i)^n (Ia)_{\overline{n}|}^{(m)} = (1+i)^n \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}} = \frac{\ddot{s}_{\overline{n}|} - n}{i^{(m)}}$$

8.03

Annuity-Immediate Increasing Once per Time Unit, Payable m^{th}ly

The present value and accumulated value of an annuity that pays $1/m$ at the end of each period in the first unit of time, $2/m$ at the end of each period in the second unit of time, and so on, until the annuity pays n/m at the end of each period in the n^{th} unit of time are:

$$(Ia)_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}} \qquad (Is)_{\overline{n}|}^{(m)} = \frac{\ddot{s}_{\overline{n}|} - n}{i^{(m)}}$$

where:

$m =$ Number of payment periods in each unit of time

The annuity-immediate in the example below makes payments monthly, and the payments increase annually.

Example 8.04

An annuity-immediate makes monthly payments at an annual rate of $100 per year during the first year, $110 per year during the second year, $120 per year during the third year, and so on, for ten years. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution

The present value of the payments can be written as follows:

$$PV_0 = \frac{100}{12}\left(v^{\frac{1}{12}} + v^{\frac{2}{12}} + \cdots v^{\frac{12}{12}}\right) + \frac{110}{12}v\left(v^{\frac{1}{12}} + v^{\frac{2}{12}} + \cdots v^{\frac{12}{12}}\right) + \cdots$$

$$+ \frac{190}{12}v^9\left(v^{\frac{1}{12}} + v^{\frac{2}{12}} + \cdots v^{\frac{12}{12}}\right)$$

This present value can also be described as the sum of a level annuity that pays 90 per year plus an increasing annuity that pays 10 per year in the first year, 20 per year in the second year, and so on:

$$\begin{aligned}
PV_0 &= 90a_{\overline{10}|}^{(12)} + 10(Ia)_{\overline{10}|}^{(12)} = 90a_{\overline{10}|}^{(12)} + 10\frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{i^{(12)}} \\
&= 90 \times \frac{1-1.005^{-120}}{0.06} + 10 \times \frac{\dfrac{1-1.005^{-120}}{1.005^{12}-1} \times 1.005^{12} - 10v^{10}}{0.06} \\
&= 90 \times 7.5061 + 10 \times \frac{7.7523 - 10(1.005)^{-120}}{0.06} = 675.5509 + 10 \times 37.5995 \\
&= \mathbf{1{,}051.55}
\end{aligned}$$

Let's again consider an annuity that makes payments m^{th}ly at a rate of 1 per unit of time in the first unit of time, 2 per unit of time in the second unit of time, and so on until payments of n per unit of time are paid in the n^{th} unit of time. This time though, let's assume that the payments are made at the beginning of each period.

			1				2								n		
Payment	$\frac{1}{m}$	…	$\frac{1}{m}$	$\frac{1}{m}$	$\frac{2}{m}$	…	$\frac{2}{m}$	$\frac{2}{m}$	$\frac{3}{m}$	…		$\frac{n-1}{m}$	$\frac{n}{m}$	…	$\frac{n}{m}$	$\frac{n}{m}$	
Time	0	$\frac{1}{m}$	…		1	…			2		…		$(n-1)$	…			n
Period	0	1	…		m	…			$2m$		…		$(n-1)m$	…			nm

The time 0 present value of this increasing annuity-due, payable m^{th}ly, is denoted by $(I\ddot{a})_{\overline{n}|}^{(m)}$.

We can accumulate each of the payments to the end of each unit of time to obtain a stream of payments that has the same present value. The following annuity-immediate has the same present value as the annuity-due shown above:

Payment		$1\,\ddot{s}_{\overline{1}|}^{(m)}$	$2\,\ddot{s}_{\overline{1}|}^{(m)}$	$3\,\ddot{s}_{\overline{1}|}^{(m)}$	…	$(n-1)\ddot{s}_{\overline{1}|}^{(m)}$	$n\ddot{s}_{\overline{1}|}^{(m)}$
Time	0	1	2	3	…	$n-1$	n

The present value is equal to the present value of a series of annuities, each of which is payable m^{th}ly:

$$\begin{aligned}
(I\ddot{a})_{\overline{n}|}^{(m)} &= v\ddot{s}_{\overline{1}|}^{(m)} + 2v^2\ddot{s}_{\overline{1}|}^{(m)} + 3v^3\ddot{s}_{\overline{1}|}^{(m)} + \cdots + nv^n\ddot{s}_{\overline{1}|}^{(m)} \\
&= \ddot{s}_{\overline{1}|}^{(m)}\left[v + 2v^2 + 3v^3 + \cdots + nv^n\right] = \ddot{s}_{\overline{1}|}^{(m)}(Ia)_{\overline{n}|} = \frac{(1+i)-1}{d^{(m)}} \times (Ia)_{\overline{n}|} \\
&= \frac{i}{d^{(m)}} \times (Ia)_{\overline{n}|} = \frac{i}{d^{(m)}} \times \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d^{(m)}}
\end{aligned}$$

The time n accumulated value is denoted by $(I\ddot{s})_{\overline{n}|}^{(m)}$, and it can be found by accumulating the present value for n units of time:

$$(I\ddot{s})_{\overline{n}|}^{(m)} = (1+i)^n (I\ddot{a})_{\overline{n}|}^{(m)} = (1+i)^n \frac{\ddot{a}_{\overline{n}|} - nv^n}{d^{(m)}} = \frac{\ddot{s}_{\overline{n}|} - n}{d^{(m)}}$$

8.04

Annuity-Due Increasing Once per Time Unit, Payable m^{th}ly

The present value and accumulated value of an annuity that pays $1/m$ at the beginning of each period in the first unit of time, $2/m$ at the beginning of each period in the second unit of time, and so on, until the annuity pays n/m at the beginning of each period in the n^{th} unit of time are:

$$(I\ddot{a})_{\overline{n}|}^{(m)} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d^{(m)}} \qquad (I\ddot{s})_{\overline{n}|}^{(m)} = \frac{\ddot{s}_{\overline{n}|} - n}{d^{(m)}}$$

where:

$m =$ Number of payment periods in each unit of time

The annuity-due in the example below makes payments monthly, and the payments increase annually.

Example 8.05

An annuity-due makes monthly payments at an annual rate of \$100 per year during the first year, \$110 per year during the second year, \$120 per year during the third year, and so on, for ten years. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution

The present value of the payments can be written as follows:

$$PV_0 = \frac{100}{12}\left(1 + v^{\frac{1}{12}} + \cdots v^{\frac{11}{12}}\right) + \frac{110}{12} v \left(1 + v^{\frac{1}{12}} + \cdots v^{\frac{11}{12}}\right) + \cdots$$
$$+ \frac{190}{12} v^9 \left(1 + v^{\frac{1}{12}} + \cdots v^{\frac{11}{12}}\right)$$

This present value can also be described as the sum of a level annuity that pays 90 per year plus an increasing annuity that pays 10 per year in the first year, 20 per year in the second year, and so on:

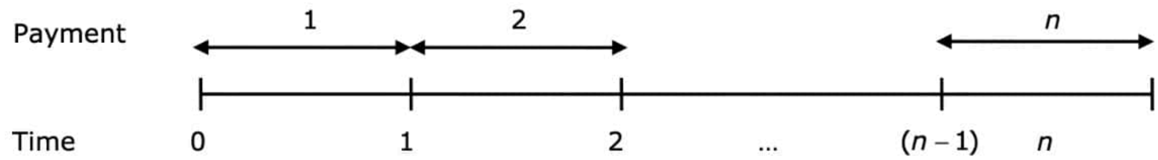
$$PV_0 = 90\ddot{a}_{\overline{10}|}^{(12)} + 10(I\ddot{a})_{\overline{10}|}^{(12)} = 90\ddot{a}_{\overline{10}|}^{(12)} + 10\frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{d^{(12)}}$$

$$= 90 \times \frac{1 - 1.005^{-120}}{(0.005/1.005) \times 12} + 10 \times \frac{\dfrac{1 - 1.005^{-120}}{1.005^{12} - 1} \times 1.005^{12} - 10v^{10}}{(0.005/1.005) \times 12}$$

$$= 90 \times 7.5437 + 10 \times \frac{7.7523 - 10(1.005)^{-120}}{0.05970} = 678.9287 + 10 \times 37.7876$$

$$= \mathbf{1{,}056.80}$$

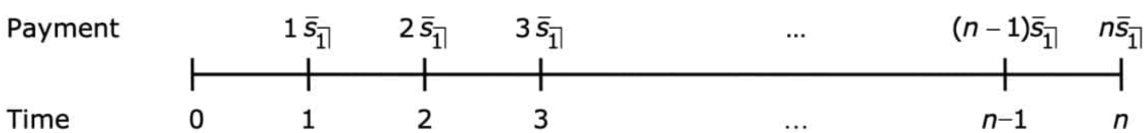
Consider an annuity that makes continuous payments at a rate of 1 per unit of time in the first unit of time, 2 per unit of time in the second unit of time, and so on until the n^{th} unit of time when payments of n per unit of time are paid. The present value of this increasing, continuous annuity is written as $(I\bar{a})_{\overline{n}|}$.



One way to find the present value of the annuity is to take the limit of an increasing annuity, payable m^{th}ly, as m approaches infinity:

$$(I\bar{a})_{\overline{n}|} = \lim_{m\to\infty}(Ia)^{(m)}_{\overline{n}|} = \lim_{m\to\infty}\frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{r}$$

Alternatively, we can accumulate each of the payment streams to the end of each unit of time to obtain a stream of payments that has the same present value. The following increasing annuity-immediate has the same present value as the continuous annuity shown above:



The present value is equal to the present value of a series of annuities, each of which is payable m^{th}ly:

$$\begin{aligned}(I\bar{a})^{(m)}_{\overline{n}|} &= v\bar{s}_{\overline{1}|} + 2v^2\bar{s}_{\overline{1}|} + 3v^3\bar{s}_{\overline{1}|} + \cdots + nv^n\bar{s}_{\overline{1}|}\\ &= \bar{s}_{\overline{1}|}\left[v + 2v^2 + 3v^3 + \cdots + nv^n\right] = \bar{s}_{\overline{1}|}(Ia)_{\overline{n}|} = \frac{(1+i)-1}{r}\times(Ia)_{\overline{n}|}\\ &= \frac{i}{r}\times(Ia)_{\overline{n}|} = \frac{i}{r}\times\frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{r}\end{aligned}$$

The notation for the accumulated value at time n is $(I\bar{s})_{\overline{n}|}$, and this accumulated value can be found by accumulating the present value for n units of time:

$$(I\bar{s})_{\overline{n}|} = (1+i)^n(I\bar{a})_{\overline{n}|} = (1+i)^n\frac{\ddot{a}_{\overline{n}|} - nv^n}{r} = \frac{\ddot{s}_{\overline{n}|} - n}{r}$$

8.05

Annuity Increasing Once per Time Unit, Payable Continuously

The present value and accumulated value of an annuity that pays continuously at a rate of 1 in the first unit of time, 2 in the second unit of time, and so until it pays at a rate of n per unit of time in the n^{th} unit of time are:

$$(I\bar{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{r} \qquad (I\bar{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{r}$$

where:

r = Force of interest

The annuity in the example below makes continuous payments that increase annually.

Example 8.06

An annuity that makes continuous payments pays at a rate of \$100 per year in the first year, \$110 per year in the second year, \$120 per year in the third year, and so on, for ten years. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution The continuously compounded interest rate is:

$$r = \ln\left(1+\frac{i^{(12)}}{12}\right)^{12} = \ln\left(1+\frac{0.06}{12}\right)^{12} = 0.05985$$

The annual payments increase by \$10 each year. The present value is:

$$90\bar{a}_{\overline{10}|} + 10(I\bar{a})_{\overline{10}|} = 90 \times \frac{1-v^{10}}{r} + 10 \times \frac{\bar{a}_{\overline{10}|} - 10v^{10}}{r}$$

$$= 90 \times \frac{1-1.005^{-10\times 12}}{0.05985} + 10 \times \frac{\dfrac{1-1.005^{-120}}{1.005^{12}-1} \times 1.005^{12} - 10 \times 1.005^{-10\times 12}}{0.05985}$$

$$= 90 \times 7.5249 + 10 \times \frac{7.7523 - 5.4963}{0.05985}$$

$$= 677.2384 + 10 \times 37.6935 = \mathbf{1{,}054.17}$$

We can adjust the increasing annuity-immediate formula to obtain the other formulas by multiplying by one of the following accumulation factors:

$$\ddot{s}_{\overline{1}|} = \frac{i}{d} \qquad s_{\overline{1}|}^{(m)} = \frac{i}{i^{(m)}} \qquad \ddot{s}_{\overline{1}|}^{(m)} = \frac{i}{d^{(m)}} \qquad \bar{s}_{\overline{1}|} = \frac{i}{r}$$

The accumulation factors are used to convert an increasing annuity into an equivalent increasing annuity-immediate by accumulating the payments to the end of each unit of time.

The following Key Concept summarizes the preceding Key Concepts in this chapter.

8.06

Annuity Increasing Once per Time Unit

Consider an annuity that pays at a rate of 1 in the first unit of time, 2 in the second unit of time, and so until it pays at a rate of n in the n^{th} unit of time.

If the annuity is an annuity-immediate that makes one payment per unit of time, then the present value and accumulated value are:

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i}$$

The present values and accumulated values for the other kinds of increasing annuities are:

$$(I\ddot{a})_{\overline{n}|} = \frac{i}{d}(Ia)_{\overline{n}|} \qquad (I\ddot{s})_{\overline{n}|} = \frac{i}{d}(Is)_{\overline{n}|}$$

$$(Ia)_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}}(Ia)_{\overline{n}|} \qquad (Is)_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}}(Is)_{\overline{n}|}$$

$$(I\ddot{a})_{\overline{n}|}^{(m)} = \frac{i}{d^{(m)}}(Ia)_{\overline{n}|} \qquad (I\ddot{s})_{\overline{n}|}^{(m)} = \frac{i}{d^{(m)}}(Is)_{\overline{n}|}$$

$$(I\bar{a})_{\overline{n}|} = \frac{i}{r}(Ia)_{\overline{n}|} \qquad (I\bar{s})_{\overline{n}|} = \frac{i}{r}(Is)_{\overline{n}|}$$

8.02 PIn Method

The PIn method is often easier to apply than the other formulas in this chapter.

Consider an annuity whose payments increase by I per unit of time. The annuity pays P_1 at the end of one unit of time. At the end of two units of time, it pays $P_1 + I$. At the end of three units of time unit, it pays $P_1 + 2I$. This continues until time n, when the annuity makes its final payment of $P_1 + (n-1)I$.

Payment		P_1	P_1+I	P_1+2I	...	$P_1+(n-1)I$
Time	0	1	2	3	...	n

We have:

$P_1 =$ First Payment
$I =$ Increment
$n =$ Number of payments

Since the primary inputs for this method are P_1, I, and n, we call this technique the PIn method.

The present value of the payments can be written as:

$$\begin{aligned}
PV_0 &= P_1 v + (P_1 + I)v^2 + (P_1 + 2I)v^3 + (P_1 + 3I)v^4 \cdots + (P + (n-1)I)v^n \\
&= P_1 a_{\overline{n}|} + Iv^2 + 2Iv^3 + 3Iv^4 + \cdots + (n-1)Iv^n \\
&= P_1 a_{\overline{n}|} + Iv\left[v^1 + 2v^2 + 3v^3 + \cdots + (n-1)v^{n-1}\right] \\
&= P_1 a_{\overline{n}|} + Iv(Ia)_{\overline{n-1}|} = P_1 a_{\overline{n}|} + Iv\frac{\ddot{a}_{\overline{n-1}|} - (n-1)v^{n-1}}{i} = P_1 a_{\overline{n}|} + I\frac{a_{\overline{n-1}|} - (n-1)v^n}{i} \\
&= P_1 a_{\overline{n}|} + \frac{I}{i}\left[a_{\overline{n-1}|} - (n-1)v^n\right] = P_1 a_{\overline{n}|} + \frac{I}{i}\left[a_{\overline{n-1}|} - (n-1)v^n + (v^n - v^n)\right] \\
&= P_1 a_{\overline{n}|} + \frac{I}{i}\left[a_{\overline{n-1}|} + v^n - (n-1+1)v^n\right] = P_1 a_{\overline{n}|} + \frac{I}{i}\left[a_{\overline{n}|} - nv^n\right] \\
&= \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n
\end{aligned}$$

Summarizing, we have the following expression for the present value of the annuity:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$

This is a useful way to express the present value because it allows us to easily find the present value of the increasing annuity using a calculator.

The example below answers the same question that was asked in Example 8.01.

Example 8.07 An annuity-immediate pays $10 at the end of 1 year, $20 at the end of 2 years, $30 at the end of 3 years, and so on, until a final payment of $120 is made at the end of 12 years. The annual effective interest rate is 9% per year. Calculate the present value of the annuity.

Solution We have:

$P_1 = 10 \qquad I = 10 \qquad n = 12$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(10 + \frac{10}{0.09}\right) \times \frac{1 - (1.09)^{-12}}{0.09} - \frac{10 \times 12}{0.09}(1.09)^{-12}$$

$$= 121.1111 \times 7.1607 - 474.0463 = \mathbf{393.20}$$

Using the BA II Plus, we have:

12 [*N*] 9 [*I/Y*] 10 [+] 10 [÷] 0.09 [=] [*PMT*]
10 [×] 12 [÷] 0.09 [=] [+/-] [*FV*] [CPT] [*PV*]
PV = –393.1971 Answer = **393.20**

The PIn method is quicker than the method we saw in Section 8.01. It also has the advantage of removing the need to put the BA II Plus in the BGN mode.

To find the accumulated value at time n, we accumulate the present value for n units of time:

$$FV_n = PV_0(1+i)^n$$

This is easily accomplished with the BA II Plus by zeroing out the value in the *PMT* register and then accumulating the value in the *PV* register.

Example 8.08 An annuity-immediate pays \$10 at the end of 1 year, \$20 at the end of 2 years, \$30 at the end of 3 years, and so on, until a final payment of \$120 is made at the end of 12 years. The annual effective interest rate is 9% per year. Calculate the accumulated value of the annuity at the end of 12 years.

Solution We have:

$$P_1 = 10 \qquad I = 10 \qquad n = 12$$

The accumulated value is:

$$FV_n = (1+i)^n\left[\left(P_1 + \frac{I}{i}\right)a_{\overline{n|}} - \frac{In}{i}v^n\right] = \left(P_1 + \frac{I}{i}\right)s_{\overline{n|}} - \frac{In}{i}$$

$$= \left(10 + \frac{10}{0.09}\right) \times \frac{(1.09)^{12} - 1}{0.09} - \frac{10 \times 12}{0.09}$$

$$= 121.1111 \times 20.1407 - 1{,}333.3333 = \mathbf{1{,}105.93}$$

Using the BA II Plus, we have:

12 [*N*] 9 [*I/Y*] 10 [+] 10 [÷] 0.09 [=] [*PMT*]
10 [×] 12 [÷] 0.09 [=] [+/-] [*FV*] [CPT] [*PV*]
PV = −393.1971
0 [*PMT*] [CPT] [*FV*]
Answer = **1,105.93**

Alternatively, we can save a few keystrokes by skipping the calculation of the present value:

12 [*N*] 9 [*I/Y*] 10 [+] 10 [÷] 0.09 [=] [*PMT*] [CPT] [*FV*]
FV = −2,439.26
[+/-] [−] 10 [×] 12 [÷] 0.09 [=]
Answer = **1,105.93**

Although it's nice to save a few keystrokes, it's also nice to use a consistent approach for these kinds of questions. Therefore, the Key Concept below suggests finding the present value and accumulating it to obtain the accumulated value.

PIn Method

8.07 The present value of an annuity-immediate that pays P_1 at time 1, P_1+I at time 2, P_1+2I, and so on until the final payment of $P_1+(n-1)I$ at time n is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$

Using the BA II Plus calculator, the calculator variables are defined as follows:

$$PMT = P_1 + \frac{I}{i} \qquad FV = -\frac{In}{i} \qquad I/Y = 100i \qquad N = n$$

With the calculator set to treat the payments as occurring at the end of each unit of time, computing the calculator's *PV* variable results in the negative of the present value:

$$CPT\ PV = -PV_0$$

The accumulated value at time n can now be found by zeroing out the calculator's *PMT* variable and then computing the calculator's *FV* variable:

$$PMT = 0$$
$$CPT\ FV = AV_n$$

The PIn method can be used in a variety of situations. For example, if we allow I to be negative, then we can use the PIn method to find the present value of a decreasing annuity.

Example 8.09 An annuity-immediate pays \$500 at the end of 1 year, \$480 at the end of 2 years, \$460 at the end of 3 years, and so on, until a final payment is made at the end of 5 years. The annual effective interest rate is 9% per year. Calculate the present value of the annuity-immediate.

Solution First, let's find the answer as follows:

$$\frac{500}{1.09} + \frac{480}{1.09^2} + \frac{460}{1.09^3} + \frac{440}{1.09^4} + \frac{420}{1.09^5} = \mathbf{1,802.60}$$

Next, we use the PIn method. We have:

$$P_1 = 500 \qquad I = -20 \qquad n = 5$$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(500 + \frac{-20}{0.09}\right) \times \frac{1-(1.09)^{-5}}{0.09} - \frac{-20\times 5}{0.09}(1.09)^{-5}$$
$$= 277.7777 \times 3.8897 + 722.1460 = \mathbf{1,802.60}$$

Using the BA II Plus, we have:

5 [*N*] 9 [*I/Y*] 500 [−] 20 [÷] 0.09 [=] [*PMT*]
20 [×] 5 [÷] 0.09 [=] [*FV*] [CPT] [*PV*]
PV = −1,802.60 Answer = **1,802.60**

The example below is the same as Example 8.02, but now we use the PIn method to obtain the answer.

Example 8.10 An annuity-immediate pays \$100 at the end of 1 month, \$110 at the end of 2 months, \$120 at the end of 3 months, and so on for 15 years.

The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution We use one month as our unit of time. We have:

$$P_1 = 100 \qquad I = 10 \qquad n = 12 \times 15 = 180$$

The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = \frac{0.06}{12} = 0.005$$

The present value is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(100 + \frac{10}{0.005}\right)a_{\overline{180}|0.005} - \frac{10 \times 180}{0.005}(1.005)^{-180}$$
$$= 2{,}100 \times 118.5035 - 360{,}000 \times 0.4075$$
$$= \mathbf{102{,}163.71}$$

Using the BA II Plus, we have:

0.5 [*I/Y*]   180 [*N*]   100 [+] 10 [÷] 0.005 [=] [*PMT*]
10 [×] 180 [÷] 0.005 [=] [+/-] [*FV*] [CPT] [*PV*]
PV = −102,163.7072
Answer = **102,163.71**

If we accumulate the payments that occur within each time unit to the end of each time unit, then we can use the PIn method for annuities that are not immediate annuities. Suppose, for example, that the payments are made at the beginning of each time unit instead of the end of each time unit, so that the increasing annuity is an annuity-due:

Payment	P_1	$P_1 + I$	$P_1 + 2I$		...	$P_1 + (n-1)I$	
Time	0	1	2	3	...	$(n-1)$	n

If we accumulate each of those payments to the end of each time unit, then we have an annuity-immediate that has the same present value as the annuity-due:

Payment		$P_1\ddot{s}_{\overline{1}|}$	$(P_1 + I)\ddot{s}_{\overline{1}|}$	$(P_1 + 2I)\ddot{s}_{\overline{1}|}$	...	$[P_1 + (n-1)I]\ddot{s}_{\overline{1}|}$
Time	0	1	2	3	...	n

The present value of the annuity-due is:

$$PV_0(\text{annuity-due}) = \ddot{s}_{\overline{1}|}\left[\left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n\right] = \frac{i}{d} \times \left[\left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n\right]$$

We can use a similar derivation to obtain the present values of other kinds of annuities as well.

PIn Method with Adjustments

8.08 Consider an annuity that pays at a rate of P_1 in the first unit of time, $P_1 + I$ in the second unit of time, $P_1 + 2I$, in the third unit of time, and so on until the final unit of time when the payment rate is $P_1 + (n-1)I$.

If the annuity is an annuity-immediate, then the present value of the annuity is:

$$PV_0(\text{annuity-immediate}) = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n$$

The other formulas for increasing annuities can be written in terms of the present value of the corresponding annuity-immediate:

$$PV_0(\text{annuity-due}) = \frac{i}{d} \times PV_0(\text{annuity-immediate})$$

$$PV_0(\text{annuity-immediate payable } m^{th}\text{ly}) = \frac{i}{i^{(m)}} \times PV_0(\text{annuity-immediate})$$

$$PV_0(\text{annuity-due payable } m^{th}\text{ly}) = \frac{i}{d^{(m)}} \times PV_0(\text{annuity-immediate})$$

$$PV_0(\text{annuity payable continuously}) = \frac{i}{r} \times PV_0(\text{annuity-immediate})$$

The example below asks the same question about an annuity-due that was originally asked in Example 8.03. We can answer the question by treating the annuity as an annuity-immediate and then adjusting that answer at the end.

Example 8.11 An annuity-due pays \$10 now, \$20 at the end of 1 year, \$30 at the end of 2 years, and so on, until a final payment of \$120 is made at the end of 11 years. The annual effective interest rate is 9% per year. Calculate the present value of the annuity.

Solution We can use the PIn method to find the present value of the corresponding annuity-immediate:

$P_1 = 10 \qquad I = 10 \qquad n = 12$

The present value of the corresponding annuity-immediate is:

$$PV_0 = \left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n = \left(10 + \frac{10}{0.09}\right) \times \frac{1-(1.09)^{-12}}{0.09} - \frac{10 \times 12}{0.09}(1.09)^{-12}$$
$$= 121.1111 \times 7.1607 - 474.0463 = 393.1971$$

The adjustment factor is the annual effective interest rate divided by the annual effective discount rate, which is equal to the one-year accumulation factor:

$$PV_0(\text{annuity-due}) = \frac{i}{d} \times PV_0(\text{annuity-immediate})$$
$$= (1+i) \times PV_0(\text{annuity-immediate})$$

The present value of the annuity-due is:

$1.09 \times 393.1971 = \mathbf{428.58}$

Using the BA II Plus, we have:

12 [*N*] 9 [*I/Y*] 10 [+] 10 [÷] 0.09 [=] [*PMT*]
10 [×] 12 [÷] 0.09 [=] [+/-] [*FV*] [CPT] [*PV*]
PV = –393.1971
[×] 1.09 [=]
Result is –428.58. Answer = **428.58**

The example below asks the same question about an annuity-immediate payable monthly that was originally asked in Example 8.04. We can answer the question by multiplying the present value of the corresponding annuity-immediate payable annually by:

$$s_{\overline{1|}}^{(m)} = \frac{i}{i^{(m)}}$$

Example 8.12 An annuity-immediate makes monthly payments at an annual rate of \$100 per year during the first year, \$110 per year during the second year, \$120 per year during the third year, and so on, for ten years. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution We can use the PIn method. We have:

$$P_1 = 100 \qquad I = 10 \qquad n = 10$$

The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = 1.005^{12} - 1 = 0.06168$$

The present value is:

$$PV_0 = s_{\overline{1|}}^{(m)}\left[\left(P_1 + \frac{I}{i}\right)a_{\overline{n|}} - \frac{In}{i}v^n\right] = \frac{i}{i^{(m)}}\left[\left(P_1 + \frac{I}{i}\right)a_{\overline{n|}} - \frac{In}{i}v^n\right]$$

$$= \frac{0.06168}{0.06}\left[\left(100 + \frac{10}{0.06168}\right)a_{\overline{10|}} - \frac{10 \times 10}{0.06168}(1.06168)^{-10}\right]$$

$$= 1.02796\left[(262.1329) \times 7.3019 - 1{,}621.3286 \times (1.06168)^{-10}\right]$$

$$= \mathbf{1{,}051.55}$$

Using the BA II Plus, we have:

1.005 [y^x] 12 [–] 1 [=] [STO] 1
[×] 100 [=] [*I/Y*]
10 [*N*] 100 [+] 10 [÷] [RCL] 1 [=] [*PMT*]
10 [×] 10 [÷] [RCL] 1 [=] [+/-] [*FV*] [CPT] [*PV*]
PV = –1,022.9414
[+/-] [×] [RCL] 1 [÷] 0.06 [=]
Answer = **1,051.55**

The example below asks the same question about an annuity-immediate payable monthly that was originally asked in Example 8.05. We can answer the question by multiplying the present value of the corresponding annuity-immediate payable annually by:

$$\ddot{s}_{\overline{1|}}^{(m)} = \frac{i}{d^{(m)}}$$

Example 8.13 An annuity-due makes monthly payments at an annual rate of \$100 per year during the first year, \$110 per year during the second year, \$120 per year during the third year, and so on, for ten years. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution We can use the PIn method. We have:

$$P_1 = 100 \qquad I = 10 \qquad n = 10$$

The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = 1.005^{12} - 1 = 0.06168$$

The present value is:

$$PV_0 = \ddot{s}_{\overline{1}|}^{(m)}\left[\left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n\right]$$
$$= \frac{i}{d^{(12)}}\left[\left(100 + \frac{10}{0.06168}\right)a_{\overline{10}|} - \frac{10 \times 10}{0.06168}(1.06168)^{-10}\right]$$
$$= \frac{0.06168}{12 \times 0.005 / 1.005}\left[(262.1329) \times 7.3019 - 1,621.3286 \times (1.06168)^{-10}\right]$$
$$= 1.03310 \times 1,022.9414$$
$$= \mathbf{1,056.80}$$

Using the BA II Plus, we have:

1.005 [y^x] 12 [–] 1 [=] [STO] 1
[×] 100 [=] [*I/Y*]
10 [*N*] 100 [+] 10 [÷] [RCL] 1 [=] [*PMT*]
10 [×] 10 [÷] [RCL] 1 [=] [+/-] [*FV*] [CPT] [*PV*]
PV = –1,022.9414
[+/-] [×] [RCL] 1 [÷] 12 [÷] 0.005 [×] 1.005 [=]
Answer = **1,056.80**

The example below asks the same question about a continuously payable annuity that was originally asked in Example 8.06. We can answer the question by multiplying the present value of the corresponding annuity-immediate payable annually by:

$$\bar{s}_{\overline{1}|}^{(m)} = \frac{i}{r}$$

Example 8.14 An annuity that makes continuous payments pays at a rate of \$100 per year in the first year, \$110 per year in the second year, \$120 per year in the third year, and so on, for ten years. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution We can use the PIn method. We find the present value of the corresponding annuity-immediate and then adjust it to find the present value of the continuous annity. We have:

$P_1 = 100$ $\qquad I = 10$ $\qquad n = 10$

The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = 1.005^{12} - 1 = 0.06168$$

The present value is:

$$PV_0 = \bar{s}_{\overline{1}|}\left[\left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n\right]$$
$$= \frac{i}{r}\left[\left(100 + \frac{10}{0.06168}\right)a_{\overline{10}|} - \frac{10 \times 10}{0.06168}(1.06168)^{-10}\right]$$
$$= \frac{0.06168}{0.05985}\left[(262.1329) \times 7.3019 - 1,621.3286 \times (1.06168)^{-10}\right]$$
$$= \mathbf{1,054.17}$$

Using the BA II Plus, we have:

1.005 [y^x] 12 [−] 1 [=] [STO] 1

[×] 100 [=] [*I/Y*]

[RCL] 1 + 1 [=] [LN] [STO] [2]

10 [*N*] 100 [+] 10 [÷] [RCL] 1 [=] [*PMT*]

10 [×] 10 [÷] [RCL] 1 [=] [+/-] [*FV*] [CPT] [*PV*]

$PV = -1{,}022.9414$

[+/-] [×] [RCL] 1 [÷] [RCL] 2 [=]

Answer = **1,054.17**

8.03 Increasing Perpetuity

If we allow n to go to infinity, then we can find the present value of an increasing perpetuity-immediate:

$$(Ia)_{\overline{\infty}|} = \lim_{n\to\infty}(Ia)_{\overline{n}|} = \lim_{n\to\infty}\frac{\ddot{a}_{\overline{n}|} - nv^n}{i} = \lim_{n\to\infty}\left[\frac{1-v^n}{di} - \frac{n}{i(1+i)^n}\right]$$

$$= \frac{1}{di} - \lim_{n\to\infty}\frac{1}{i(1+i)^n \ln(1+i)} = \frac{1}{di}$$

The present values of an increasing perpetuity-due, an increasing perpetuity-immediate payable m^{th}ly, an increasing perpetuity-due payable m^{th}ly, and an increasing perpetuity payable continuously are found below:

$$(I\ddot{a})_{\overline{\infty}|} = \frac{i}{d}(Ia)_{\overline{\infty}|} = \frac{i}{d}\times\frac{1}{di} = \frac{1}{d^2}$$

$$(Ia)^{(m)}_{\overline{\infty}|} = \frac{i}{i^{(m)}}(Ia)_{\overline{\infty}|} = \frac{i}{i^{(m)}}\times\frac{1}{di} = \frac{1}{d\times i^{(m)}}$$

$$(I\ddot{a})^{(m)}_{\overline{\infty}|} = \frac{i}{d^{(m)}}(Ia)_{\overline{\infty}|} = \frac{i}{d^{(m)}}\times\frac{1}{di} = \frac{1}{d\times d^{(m)}}$$

$$(I\bar{a})_{\overline{\infty}|} = \frac{i}{r}(Ia)_{\overline{\infty}|} = \frac{i}{r}\times\frac{1}{di} = \frac{1}{dr}$$

Increasing Perpetuities

8.09

Consider a perpetuity that pays at a rate of 1 in the first unit of time, 2 in the second unit of time, and so until it pays at a rate of n in the n^{th} unit of time. The present value of the annuity depends on the timing of the payments within each unit of time:

$$(Ia)_{\overline{\infty}|} = \frac{1}{di}$$

$$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

$$(Ia)^{(m)}_{\overline{\infty}|} = \frac{1}{d\times i^{(m)}}$$

$$(I\ddot{a})^{(m)}_{\overline{\infty}|} = \frac{1}{d\times d^{(m)}}$$

$$(I\bar{a})_{\overline{\infty}|} = \frac{1}{dr}$$

In the example below, continuous payments are made at an annual rate that increases forever.

Example 8.15

An annuity that makes continuous payments pays at a rate of \$100 per year in the first year, \$110 per year in the second year, \$120 per year in the third year, and so on forever. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution

The continuously compounded interest rate is:

$$r = \ln\left(1 + \frac{i^{(12)}}{12}\right)^{12} = \ln\left(1 + \frac{0.06}{12}\right)^{12} = 0.05985$$

The annual effective discount rate is:

$$d = 1 - e^{-r} = 1 - e^{-0.05985} = 0.05809$$

$$r = \ln\left(1 + \frac{i^{(12)}}{12}\right)^{12} = \ln\left(1 + \frac{0.06}{12}\right)^{12} = 0.05985$$

The annual payments increase by 10 each year. The present value is:

$$90\bar{a}_{\overline{\infty}|} + 10(I\bar{a})_{\overline{\infty}|} = 90 \times \frac{1}{r} + 10 \times \frac{1}{d \times r}$$

$$= 90 \times \frac{1}{0.05985} + 10 \times \frac{1}{0.05809 \times 0.05985} = \mathbf{4,379.79}$$

The formula underlying the PIn method allows us to derive a convenient formula for a perpetuity that pays P_1 in one unit of time and increases by I thereafter:

$$PV_0 = \lim_{n\to\infty}\left[\left(P_1 + \frac{I}{i}\right)a_{\overline{n}|} - \frac{In}{i}v^n\right] = \left[\left(P_1 + \frac{I}{i}\right)\frac{1}{i}\right] = \frac{P_1}{i} + \frac{I}{i^2}$$

To find the value of an increasing perpetuity-due, we include the value of a payment that occurs at time 0, P_0. The present value at time 0 of an increasing perpetuity-due that pays P_0 at time 0, P_1 at the time 1, $(P_1 + I)$ at time 2, $(P_1 + 2I)$ at time 3, and so on is:

$$PV_0 = P_0 + \frac{P_1}{i} + \frac{I}{i^2}$$

8.10

Increasing Perpetuity-Due

The present value at time 0 of an increasing perpetuity-due that pays P_0 at time 0, P_1 at the time 1, $(P_1 + I)$ at time 2, $(P_1 + 2I)$ at time 3, and so on is:

$$PV_0 = P_0 + \frac{P_1}{i} + \frac{I}{i^2}$$

The formula in the Key Concept above can be used to calculate the value of a perpetuity-immediate by setting P_0 equal to 0.

The example below is an annuity-due, so we set P_0 equal to the first payment.

Example 8.16

An annuity-due makes a payment of \$100 now, \$110 in one year, \$120 in two years, and so on forever. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution

The annual effective interest rate is:

$$i = \left(1 + \frac{i^{(12)}}{12}\right)^{12} - 1 = \left(1 + \frac{0.06}{12}\right)^{12} - 1 = 0.06168$$

The present value of the annuity is:

$$PV_0 = P_0 + \frac{P_1}{i} + \frac{I}{i^2} = 100 + \frac{110}{0.06168} + \frac{10}{0.6168^2} = \mathbf{4{,}512.17}$$

8.04 Decreasing Annuity

Consider an annuity-immediate whose payments decrease by 1 per unit of time. The annuity pays n at the end of one unit of time. At the end of the second unit of time, it pays $(n-1)$. At the end of the third time unit, it pays $(n-2)$. This continues until time n, when the annuity makes its final payment of 1.

Payment		n	$n-1$	$n-2$	...	2	1
Time	0	1	2	3	...	$n-1$	n

We use $(Da)_{\overline{n}|}$ to denote the present value of this annuity-immediate:

$$(Da)_{\overline{n}|} = nv + (n-1)v^2 + (n-2)v^3 + \cdots + 2v^{n-1} + v^n$$

Let's multiply the expression above by $(1+i)$ and then subtract the original expression from both sides:

$$(1+i)(Da)_{\overline{n}|} = n + (n-1)v^1 + (n-2)v^2 + \cdots + 2v^{n-2} + v^{n-1}$$

$$(Da)_{\overline{n}|} = nv + (n-1)v^2 + (n-2)v^3 + \cdots + 2v^{n-1} + v^n$$

$$(1+i)(Da)_{\overline{n}|} - (Da)_{\overline{n}|} = n - v - v^2 - \cdots - v^{n-1} - v^n$$

$$i(Da)_{\overline{n}|} = n - a_{\overline{n}|}$$

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i}$$

The accumulated value of the decreasing annuity-immediate at time n is denoted by $(Ds)_{\overline{n}|}$ and it is found below by accumulating the present value for n years:

$$(Ds)_{\overline{n}|} = (1+i)^n (Da)_{\overline{n}|} = (1+i)^n \frac{n - a_{\overline{n}|}}{i} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

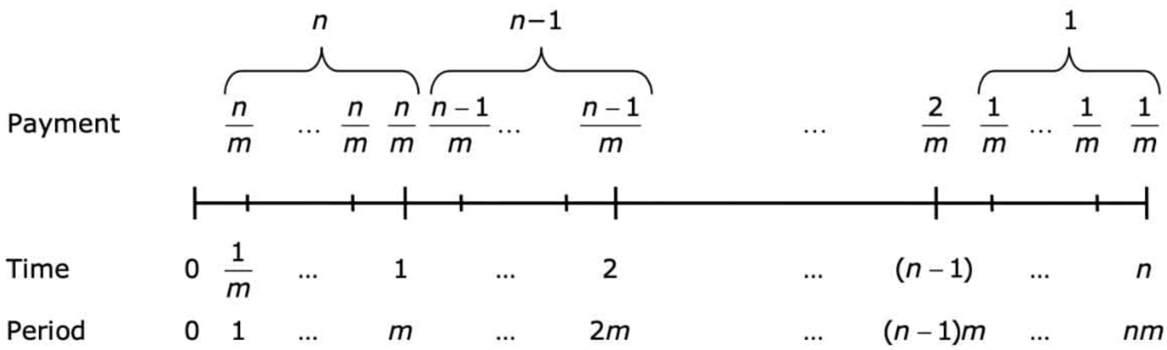
If the payments of a decreasing annuity occur at the beginning of each unit of time, then the annuity is a decreasing annuity-due. We use $(D\ddot{a})_{\overline{n}|}$ to denote the present value of a decreasing annuity-due:

Payment	n	$n-1$	$n-2$	...	2	1	
Time	0	1	2	...	$n-2$	$n-1$	n

By accumulating the payments to the end of each unit of time, we can find the present value:

$$(D\ddot{a})_{\overline{n}|} = \ddot{s}_{\overline{1}|}(Da)_{\overline{n}|} = \frac{i}{d} \times \frac{n - a_{\overline{n}|}}{i} = \frac{n - a_{\overline{n}|}}{d}$$

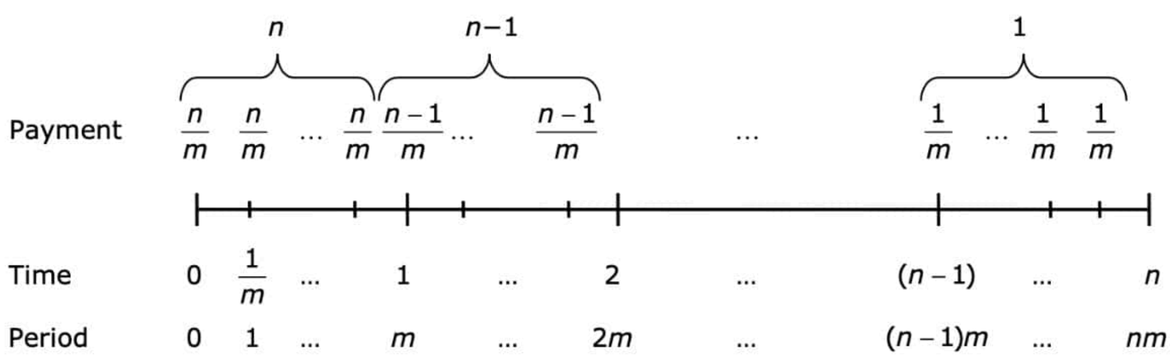
We use $(Da)_{\overline{n}|}^{(m)}$ to denote the present value of a decreasing annuity-immediate payable m^{th}ly:



By accumulating the payments to the end of each unit of time, we can find the present value:

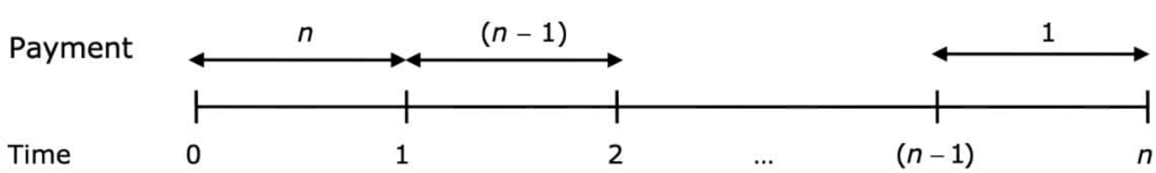
$$(Da)_{\overline{n}|}^{(m)} = s_{\overline{1}|}^{(m)}(Da)_{\overline{n}|} = \frac{i}{i^{(m)}} \times \frac{n - a_{\overline{n}|}}{i} = \frac{n - a_{\overline{n}|}}{i^{(m)}}$$

We use $(D\ddot{a})_{\overline{n}|}^{(m)}$ to denote the present value of a decreasing annuity-due payable m^{th}ly:



$$(D\ddot{a})_{\overline{n}|}^{(m)} = \ddot{s}_{\overline{1}|}^{(m)}(Da)_{\overline{n}|} = \frac{i}{d^{(m)}} \times \frac{n - a_{\overline{n}|}}{i} = \frac{n - a_{\overline{n}|}}{d^{(m)}}$$

We use $(D\bar{a})_{\overline{n}|}$ to denote the present value of a decreasing annuity payable continuously:



By accumulating the payments to the end of each unit of time, we can find the present value:

$$(D\bar{a})_{\overline{n}|} = \bar{s}_{\overline{1}|}(Da)_{\overline{n}|} = \frac{i}{r} \times \frac{n - a_{\overline{n}|}}{i} = \frac{n - a_{\overline{n}|}}{r}$$

The Key Concept below summarizes the formulas for decreasing annuities.

8.11

Decreasing Annuity

Consider a decreasing annuity that pays at a rate of n in the first unit of time, $(n-1)$ in the second unit of time, and so on until it pays at a rate of 1 in the n^{th} unit of time.

If the annuity is an annuity-immediate, then the present value at time 0 and accumulated value at time n are:

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \qquad (Ds)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

The present values and accumulated values for the other decreasing annuities are:

$$(D\ddot{a})_{\overline{n}|} = \frac{i}{d}(Da)_{\overline{n}|} \qquad (D\ddot{s})_{\overline{n}|} = \frac{i}{d}(Ds)_{\overline{n}|}$$

$$(Da)^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}}(Da)_{\overline{n}|} \qquad (Ds)^{(m)}_{\overline{n}|} = \frac{i}{i^{(m)}}(Ds)_{\overline{n}|}$$

$$(D\ddot{a})^{(m)}_{\overline{n}|} = \frac{i}{d^{(m)}}(Da)_{\overline{n}|} \qquad (D\ddot{s})^{(m)}_{\overline{n}|} = \frac{i}{d^{(m)}}(Ds)_{\overline{n}|}$$

$$(D\bar{a})_{\overline{n}|} = \frac{i}{r}(Da)_{\overline{n}|} \qquad (D\bar{s})_{\overline{n}|} = \frac{i}{r}(Ds)_{\overline{n}|}$$

The example below asks the same question that is asked in Example 8.09. In Example 8.09, we answer the question using the PIn method.

Example 8.17

An annuity-immediate pays \$500 at the end of 1 year, \$480 at the end of 2 years, \$460 at the end of 3 years, and so on, until a final payment is made at the end of 5 years. The annual effective interest rate is 9% per year. Calculate the present value of the annuity-immediate.

Solution

The present value of the annuity-immediate is:

$$\begin{aligned}
&500v + 480v^2 + 460v^3 + 440v^4 + 420v^5 \\
&\quad = 400a_{\overline{5}|} + 100v + 80v^2 + 60v^3 + 40v^4 + 20v^5 \\
&\quad = 400a_{\overline{5}|} + 20\left(5v + 4v^2 + 3v^3 + 2v^4 + v^5\right) \\
&\quad = 400a_{\overline{5}|} + 20(Da)_{\overline{5}|} = 400 \times \frac{1 - 1.09^{-5}}{0.09} + 20 \times \frac{5 - a_{\overline{5}|}}{0.09} \\
&\quad = 400 \times 3.8897 + 20 \times \frac{5 - 3.8897}{0.09} = 400 \times 3.8897 + 20 \times 12.3372 \\
&\quad = \mathbf{1{,}802.60}
\end{aligned}$$

Many students find the PIn method to be an easier way to answer these kinds of questions.

Example 8.18

An annuity-due pays \$500 now, \$480 at the end of 1 year, \$460 at the end of 2 years, and so on, until a final payment is made at the end of 4 years. The annual effective interest rate is 9% per year. Calculate the present value of the annuity-due.

Solution The present value of the annuity-due is:

$$\begin{aligned}
&500+480v+460v^2+440v^3+420v^4\\
&\quad=400\ddot{a}_{\overline{5}|}+100+80v+60v^2+40v^3+20v^4\\
&\quad=400\ddot{a}_{\overline{5}|}+20\left(5+4v+3v^2+2v^3+v^4\right)\\
&\quad=400\ddot{a}_{\overline{5}|}+20(D\ddot{a})_{\overline{5}|}=400\times\frac{1-1.09^{-5}}{0.09}\times1.09+20\times\frac{5-a_{\overline{5}|}}{0.09}\times1.09\\
&\quad=400\times4.2397+20\times\frac{5-3.8897}{0.09}\times1.09=400\times4.2397+20\times13.4476\\
&\quad=\mathbf{1,964.84}
\end{aligned}$$

Alternatively, we can use the PIn method to find the present value. When using the PIn method, the first payment occurs in 1 year. Therefore let's find the value of the final 4 payments, and then we can add the $500 payment to the present value of the final 4 payments.

Using the BA II Plus, we have:

4 [*N*] 9 [*I/Y*] 480 [-] 20 [÷] 0.09 [=] [*PMT*]
20 [×] 4 [÷] 0.09 [=] [*FV*] [CPT] [*PV*]
PV = −1,464.8391
[+/-] [+] 500 [=] Answer = **1,964.84**

8.05 Questions

Question 8.01

An annuity pays 50 in 1 year, 55 in 2 years, 60 in 3 years, and so on until a final payment of 150 in 21 years.

The annual effective interest rate is 5%.

Calculate the present value of the annuity.

A 753.65 B 1,054.67 C 1,105.29 D 1,169.39 E 2,676.95

Question 8.02

An annuity pays 50 in 6 months, 55 in 1.5 years, 60 in 2.5 years, and so on until a final payment of 150 in 20.5 years.

The annual effective interest rate is 5%.

Calculate the present value of the annuity.

A 1,141.21 B 1,169.39 C 1,198.27 D 1,227.86 E 3,409.96

Question 8.03

An annuity pays 50 in 1 year, 55 in 2 years, 60 in 3 years, and so on until a final payment of 150 in 21 years.

The annual effective interest rate is 5%.

Calculate the accumulated value of the annuity at the end of 21 years.

A 2,959.89 B 3,079.29 C 3,107.89 D 3,257.89 E 3,338.34

Question 8.04

A 10-year annuity-immediate makes monthly payments at a rate of $60 per year in the first year, $120 per year in the second year, $180 per year in the third year, and so on. The nominal annual interest rate is 6% compounded monthly.

Calculate the accumulated value of the annuity at the end of 10 years.

A 2,256 B 2,267 C 3,884 D 3,993 E 4,105

Question 8.05

A 20-year annuity-due pays 2 at the beginning of each month during the 1st year, 4 at the beginning of each month during the 2nd year, 6 at the beginning of each month during the 3rd year, and so on. The nominal annual interest rate is 12% compounded monthly.

Calculate the present value of the annuity-due.

A 1,116.05 B 1,179.53 C 1,246.62 D 1,259.08 E 1,404.72

Question 8.06

Alex receives level payments at the beginning of each year for 5 years. He deposits the payments into a fund. The deposits earn an annual effective interest rate of 6%, which is paid at the end of each year. The interest is immediately reinvested at an annual effective interest rate of 3%.

At the end of 5 years, the accumulated value in the fund is 800.

Calculate the amount of each level payment.

A 134.75 B 137.74 C 142.39 D 150.42 E 150.67

Question 8.07

Jane receives level payments at the beginning of each year for 15 years. She deposits the payments into a fund. The deposits earn an annual effective interest rate of 7%, which is paid at the end of each year. The interest is immediately reinvested at an annual effective interest rate of 4%.

At the end of 15 years, the accumulated value in the fund is 25,000.

Calculate the amount of each level payment.

A 697.85 B 767.28 C 992.34 D 1,050.81 E 1,113.91

Question 8.08

A bond with 9 years remaining until maturity makes semiannual coupon payments that increase by X every six months. The most recent coupon payment was just made in the amount of 30. The bond's redemption value of 1,000 will be paid upon maturity.

The yield of the bond is 8% compounded semiannually, and its price today is 1,300.

Calculate X.

A 3.84 B 3.99 C 4.52 D 7.65 E 11.08

Question 8.09

An annuity consists of a series of 75 payments starting at 200 at the end of the first year and increasing by 4 each year thereafter.

The annual effective interest rate is 8%.

Calculate the present value of the annuity.

A 3,050.75 B 3,053.75 C 3,103.60 D 3,153.44 E 3,351.81

Question 8.10

A loan of 15,000 is repaid with 25 payments made at the end of each year for 25 years. The first payment is 100, the second payment is 200, and so on until the tenth payment of 1,000. The final 15 payments are X per year.

The annual effective interest rate is 3% per year.

Calculate X.

A 880.90 B 1,183.85 C 1,372.41 D 1,656.79 E 1,920.67

Question 8.11

The present value of a 6-year annuity is X.

The annuity pays 3 at the end of the first month, 6 at the end of the second month, and for each month thereafter, the payment increases by 3.

The annual nominal interest rate is 7% compounded quarterly.

Calculate X.

A 3,387.31 B 4,820.31 C 5,820.62 D 5,991.24 E 6,167.40

Question 8.12

Joel purchases a perpetuity-due for 3,150 with annual payments of 150. At the same price and interest rate, Ellen purchases an annuity-immediate with 15 annual payments that begin at amount P and increase by 20 each year thereafter.

Calculate P.

A 161.00 B 175.44 C 181.53 D 183.96 E 200.07

Question 8.13

A loan is to be repaid with a 20-year annuity that makes its first payment in the amount of 500 in five years. Thereafter, the remaining 19 payments increase by 250 per year.

The annual effective interest rate is 6% per year.

Calculate the amount of the loan.

A 19,358.17 B 20,519.66 C 20,558.50 D 20,581.41 E 21,816.30

Question 8.14

The present value of a 6-year annuity is X.

The annuity pays 5 at the end of the first month, 10 at the end of the second month, and for each month thereafter the payment increases by 5.

The annual nominal interest rate is 7% compounded quarterly.

Calculate X.

A 9,927.82 B 9,985.40 C 9,993.11 D 10,043.31 E 10,050.90

Question 8.15

A 10-year increasing annuity makes continuous payments at a rate of \$50 per year in the 1st year, \$100 per year in the 2nd year, \$150 per year in the 3rd year, and so on. No payments are made after 10 years have elapsed.

The annual continuously compounded interest rate is 7%.

Calculate the present value of the annuity.

A 1,651.28 B 1,703.25 C 1,710.45 D 1,762.19 E 1,771.74

Question 8.16

An annuity pays 100 at the end of 3 months, 125 at the end of 6 months, 150 at the end of 9 months, and so on until it makes a final payment at the end of 7 years.

The annual effective discount rate is 8%.

Calculate the future value of the annuity at the end of 7 years.

A 14,877 B 14,928 C 15,190 D 15,218 E 15,510

Question 8.17

A 5-year annuity-immediate pays 10 at the end of each month for 6 months, 20 at the end of each month for the next 6 months, 30 at the end of the next 6 months, and so on until a final payment of 100 is made at the end of 5 years.

The nominal annual interest rate is 6% compounded monthly.

Calculate the present value of the annuity-immediate.

A 2,200.14 B 2,400.63 C 2,650.28 D 2,683.63 E 2,717.40

Question 8.18

A perpetuity-due pays 1 per year, and its present value is 26.

An increasing perpetuity-due pays X immediately, $2X$ in one year, $3X$ in two years, and so on. The present value of the increasing perpetuity-due is 4,732.

What is the amount of the 8th payment made by the increasing perpetuity-due?

A 56 B 58 C 61 D 63 E 64

Question 8.19

A perpetuity-immediate pays 300 at the end of the first year, 325 at the end of the second year, 350 at the end of the third year, and so on. The present value of the perpetuity-immediate is 21,315.

The annual effective rate of interest is i.

Calculate i.

A 2.1% B 2.8% C 4.1% D 4.2% E 8.4%

Question 8.20

A perpetuity-due pays 275 now, and each subsequent payment is greater than its predecessor by 25. The present value of the perpetuity-due is 21,590.

The annual effective rate of interest is i.

Calculate i.

A 2.1% B 2.8% C 4.0% D 4.1% E 4.2%

Question 8.21

A perpetuity pays 1 at the end of year 2, 2 at the end of year 3, and so on, until it makes a payment of n at time $(n+1)$. At time $(n+2)$ and at the end of each year thereafter, the perpetuity makes a payment of n.

The annual effective interest rate is 7.5%, and the present value of the perpetuity is 144.09.

Calculate n.

A 22 B 23 C 24 D 25 E 26

Question 8.22

An annuity pays 50 in 1 year, 45 in 2 years, 40 in 3 years, and so on until a final payment of 5 in 10 years.

The annual effective interest rate is 5%.

Calculate the present value of the annuity.

A 189.22 B 198.68 C 216.98 D 227.83 E 239.22

Question 8.23

An annuity pays 50 in 1 year, 45 in 2 years, 40 in 3 years, and so on until a final payment of 5 in 10 years.

The annual effective interest rate is 5%.

Calculate the accumulated value of the annuity at the end of 10 years.

A 227.83 B 257.79 C 308.22 D 353.43 E 371.11

Question 8.24

A 20-year annuity-due's first payment is 500. Each subsequent payment decreases by 25 per year. Its present value is X.

The annual effective interest rate is 6%.

Calculate X.

A 3,353.02 B 3,554.20 C 3,767.45 D 3,993.50 E 4,233.11

Question 8.25

A perpetuity-due pays 1 per year, and its present value is 21.

A decreasing annuity-immediate pays X in one year, $(X - 1)$ in two years, $(X - 2)$ in three years, and so on until a final payment of $(X - 9)$ in ten years. The present value of the decreasing annuity-immediate is 68.73.

What is the amount of the 8th payment made by the decreasing annuity-immediate?

A 6 B 7 C 13 D 20 E 27

Question 8.26

An annuity-immediate makes payments of 30 for 20 years, and then the payments decrease by 1 per year for 29 years.

The annual effective interest rate is 9%.

Calculate the present value of the annuity-immediate.

A 225.87 B 229.49 C 285.44 D 307.61 E 311.13

Question 8.27

A 5-year annuity-immediate pays 25 at the end of each quarter during the 1st year, 20 at the end of each quarter during the 2nd year, 15 at the end of each quarter during the 3rd year, 10 at the end of each quarter during the 4th year, and finally 5 at the end of each quarter during the 5th year.

The nominal annual interest rate compounded quarterly is 8%.

Calculate the present value of the annuity-immediate.

A 243.18 B 245.66 C 250.57 D 255.58 E 258.19

Question 8.28

The present value of a 5-year annuity is X.

The annuity pays 600 at the end of the first month, 590 at the end of the second month, and for each month thereafter the payment decreases by 10.

The annual nominal interest rate is 3% compounded monthly.

Calculate X.

A 16,402.37 B 16,834.05 C 16,845.29 D 17,390.57 E 17,402.37

Question 8.29

A 10-year decreasing annuity makes continuous payments at a rate of \$500 per year in the 1st year, \$450 per year in the 2nd year, \$400 per year in the 3rd year, and so on. No payments are made after 10 years have elapsed.

The annual continuously compounded interest rate is 7%.

Calculate the present value of the annuity.

A 1,965.61 B 2,035.21 C 2,036.04 D 2,108.13 E 2,183.67

Question 8.30

An annuity-immediate makes payments of 30 for 20 years, and then the payments decrease by 2 per year for 14 years.

The annual effective interest rate is 9%.

Calculate the present value of the annuity-immediate.

A 249.22 B 295.36 C 298.50 D 317.29 E 325.36

Question 8.31

Fund A has an initial balance of 1,000 and earns interest at an annual effective interest rate of 7%. At the end of each year, the interest credited to Fund A is withdrawn and deposited into Fund B. At the same time, at the end of each year, 100 is withdrawn from Fund A and deposited into Fund B. At the end of 10 years, the balance in Fund A falls to zero.

The annual effective interest rate earned by Fund B is 5%.

Calculate the accumulated value in Fund B at the end of year 10.

A 1,091.13 B 1,576.75 C 1,777.34 D 4,256.34 E 6,933.12

Chapter 9: Continuously Payable Varying Payments

9.01 Continuously Varying Payment Stream

Suppose that a financial instrument makes m^{th}ly payments at the end of each period at a rate of Pmt_t for n time units, and the continuously compounded interest rate from time $(t-1/m)$ to time t is denoted by r_t:

$$Pmt_t = \text{Time } t \text{ rate of payment per unit of time}$$

$$\frac{Pmt_t}{m} = \text{Payment amount at time } t$$

$$r_t = \begin{cases} \text{Continuously compounded interest rate} \\ \text{applicable from time } \left(t-\dfrac{1}{m}\right) \text{ to time } t \end{cases}$$

Payment		$\frac{Pmt_{\frac{1}{m}}}{m}$	$\frac{Pmt_{\frac{2}{m}}}{m}$	...	$\frac{Pmt_k}{m}$	...	$\frac{Pmt_{\frac{nm-1}{m}}}{m}$	$\frac{Pmt_n}{m}$
Time	0	$\frac{1}{m}$	$\frac{2}{m}$	...	k	...	$\frac{nm-1}{m}$	n
Period	0	1	2	...	km	...	$nm-1$	nm

The present value of this stream of payments at time 0 is:

$$\begin{aligned} PV_0 &= \frac{1}{m} Pmt_{\frac{1}{m}} e^{-r_{\frac{1}{m}} \times \frac{1}{m}} + \frac{1}{m} Pmt_{\frac{2}{m}} e^{-r_{\frac{1}{m}} \times \frac{1}{m} - r_{\frac{2}{m}} \times \frac{1}{m}} + \cdots + \frac{1}{m} Pmt_{\frac{nm}{m}} e^{-\left(r_{\frac{1}{m}} + r_{\frac{2}{m}} + \cdots + r_{\frac{nm}{m}}\right) \times \frac{1}{m}} \\ &= \frac{1}{m} Pmt_{\frac{1}{m}} e^{-r_{\frac{1}{m}} \times \frac{1}{m}} + \frac{1}{m} Pmt_{\frac{2}{m}} e^{-\sum_{k=1}^{2}\left(r_{\frac{k}{m}} \times \frac{1}{m}\right)} + \cdots + \frac{1}{m} Pmt_{\frac{nm}{m}} e^{-\sum_{k=1}^{nm}\left(r_{\frac{k}{m}} \times \frac{1}{m}\right)} \\ &= \sum_{j=1}^{nm}\left[Pmt_{\frac{j}{m}} e^{-\sum_{k=1}^{j}\left(r_{\frac{k}{m}} \times \frac{1}{m}\right)} \times \frac{1}{m}\right] \end{aligned}$$

As m increases to infinity, this can be written as:

$$PV_0 = \lim_{m \to \infty} \sum_{j=1}^{nm}\left[Pmt_{\frac{j}{m}} e^{-\sum_{k=1}^{j}\left(r_{\frac{k}{m}} \times \frac{1}{m}\right)} \times \frac{1}{m}\right] = \int_0^n Pmt_t e^{-\int_0^t r_s ds} dt$$

Let's generalize this further. Instead of beginning at time 0 and ending at time n, suppose that the beginning of the first period occurs at time a, and the end of the final time period occurs at time b:

$$PV_a = \int_a^b Pmt_t e^{-\int_a^t r_s ds} dt$$

The accumulated value at time b is the present value accumulated from time a to time b:

$$AV_b = e^{\int_a^b r_s ds} \times PV_a = e^{\int_a^b r_s ds} \int_a^b Pmt_t e^{-\int_a^t r_s ds} dt = \int_a^b Pmt_t e^{\int_a^b r_s ds} e^{-\int_a^t r_s ds} dt$$

$$= \int_a^b Pmt_t e^{\int_t^b r_s ds} dt$$

The current value of the stream of payments can be found at any time k by accumulating from time a to time k:

$$CV_k = e^{\int_a^k r_s ds} \times PV_a = e^{\int_a^k r_s ds} \int_a^b Pmt_t e^{-\int_a^t r_s ds} dt = \int_a^b Pmt_t e^{\int_a^k r_s ds} e^{-\int_a^t r_s ds} dt$$

$$= \int_a^b Pmt_t e^{\int_t^k r_s ds} dt$$

The current value at time a is equal to the present value at time a, and the current value at time b is equal to the accumulated value at time b:

$$CV_a = \int_a^b Pmt_t e^{\int_t^a r_s ds} dt = \int_a^b Pmt_t e^{-\int_a^t r_s ds} dt = PV_a$$

$$CV_b = \int_a^b Pmt_t e^{\int_t^b r_s ds} dt = AV_b$$

Continuously Varying Payment Stream and Force of Interest

9.01 The present value at time a and the accumulated value at time b of payments made continuously at a rate of Pmt_t per unit of time when the force of interest is r_t per unit of time are:

$$PV_a = \int_a^b Pmt_t e^{-\int_a^t r_s ds} dt$$

$$AV_b = \int_a^b Pmt_t e^{\int_t^b r_s ds} dt$$

The payment rate in the example below is a function of time.

Example 9.01 A 10-year annuity makes continuous payments at an annual rate of $(100 + 10t)$ for $0 \leq t \leq 10$. The annual interest rate compounded monthly is 6%. Calculate the present value of the annuity.

Solution The continuously compounded interest rate is:

$$r = \ln\left(1 + \frac{i^{(12)}}{12}\right)^{12} = \ln\left(1 + \frac{0.06}{12}\right)^{12} = 0.05985$$

The present value is equal to the current value at time 0:

$$PV_0 = \int_0^{10} Pmt_t e^{-\int_0^t r_s ds} dt = \int_0^{10} (100 + 10t) e^{-\int_0^t r ds} dt$$

$$= \int_0^{10} (100 + 10t) e^{-rt} dt$$

Let's use the following expression to integrate by parts:

$$\int u dv = uv - \int v du$$

We can set u and v as follows:

$$u = 100 + 10t \qquad dv = e^{-rt}dt$$

$$du = 10dt \qquad v = -\frac{e^{-rt}}{r}$$

We have:

$$\int (100 + 10t)e^{-rt}dt = -\frac{(100 + 10t)e^{-rt}}{r} - \int -\frac{10e^{-rt}}{r}dt = -\frac{(100 + 10t)e^{-rt}}{r} - \frac{10e^{-rt}}{r^2}$$

The present value is:

$$PV_0 = \int_0^{10} (100 + 10t)e^{-rt}dt = \left[-\frac{(100 + 10t)e^{-rt}}{r} - \frac{10e^{-rt}}{r^2}\right]_0^{10}$$

$$= -\frac{200e^{-10r}}{r} - \frac{10e^{-10r}}{r^2} + \left[\frac{100}{r} + \frac{10}{r^2}\right] = \frac{100 - 200e^{-10r}}{r} + \frac{10 - 10e^{-10r}}{r^2}$$

$$= \frac{100 - 200e^{-10\times 0.05985}}{0.05985} + \frac{10 - 10e^{-10\times 0.05985}}{0.05985^2}$$

$$= -165.8557 + 1{,}257.2779 = \mathbf{1{,}091.42}$$

The v used when integrating by parts in the example above is not equal to the discount factor for one time unit. There are two conventions for using v, so be careful not to confuse them. The example uses the second bullet point below:

- *v is the discount factor for one time unit,* $v = \frac{1}{1+i}$
- *v is used when integrating by parts.*

In the example below, both the rate of payment and the continuously interest rate are functions of time.

Example 9.02

A financial instrument makes continuous payments for 15 years at an annual rate of $(100 + 10t)$ for $0 \le t \le 15$. The continuously compounded interest rate over that time interval is $(0.05 + 0.005t)$.

Calculate the present value of the annuity.

Solution

The present value is:

$$PV_0 = \int_0^{15} Pmt_t e^{-\int_0^t r_s ds}dt = \int_0^{15} (100 + 10t)e^{-\int_0^t (0.05+0.005s)ds}dt$$

Let's use the following substitution:

$$u = \int_0^t (0.05 + 0.005s)ds = 0.05t + 0.0025t^2$$

$$du = (0.05 + 0.005t)dt$$

$$2{,}000du = (100 + 10t)dt$$

We have:

$$PV_0 = \int_0^{15} (100+10t)e^{-\int_0^t (0.05+0.005s)ds}dt = \int_{t=0}^{t=15} 2{,}000e^{-u}du = -2{,}000e^{-u}\Big|_{t=0}^{t=15}$$

$$= -2{,}000e^{-0.05t-0.0025t^2}\Big|_0^{15} = -2{,}000\left[e^{-1.3125}-1\right] = \mathbf{1{,}461.71}$$

The integration in these kinds of problems is not very difficult when, as in the example above, the payment rate is a multiple of the continuously compounded interest rate.

Example 9.03 A financial instrument makes continuous payments for 15 years at an annual rate of $(100+10t)$ for $0 \le t \le 15$. The continuously compounded interest rate over that time interval is $(0.05+0.005t)$.

Calculate the accumulated value of the annuity at the end of 15 years.

Solution The future value is:

$$FV_{15} = \int_0^{15} Pmt_t e^{\int_t^{15} r_s ds}dt = \int_0^{15} (100+10t)e^{\int_t^{15}(0.05+0.005s)ds}dt$$

Let's use the following substitution:

$$u = \int_t^{15}(0.05+0.005s)ds = 1.3125 - 0.05t - 0.0025t^2$$

$$du = -(0.05+0.005t)dt \quad \text{so} \quad 2{,}000du = -(100+10t)dt$$

We have:

$$FV_{15} = \int_0^{15} (100+10t)e^{\int_t^{15}(0.05+0.005s)ds}dt = \int_{t=0}^{t=15} -2{,}000e^{u}du = -2{,}000e^{u}\Big|_{t=0}^{t=15}$$

$$= -2{,}000e^{1.3125-0.05t-0.0025t^2}\Big|_0^{15} = -2{,}000\left[e^{1.3125-0.75-0.5625} - e^{1.3125}\right]$$

$$= -2{,}000\left[1-e^{1.3125}\right] = \mathbf{5{,}430.90}$$

In the example below, the rate of payment is a multiple of the inverse of the continuously compounded interest rate.

Example 9.04 Beginning at time 0, payments are made to an account at a rate of $(6x+tx)$.

Interest is credited at the following force of interest:

$$\delta_t = \frac{1}{6+t}, \quad \text{where } 0 \le t \le 25$$

At time 25, the accumulated value of the account is 7,750.

What is the value of x?

Solution The equation of value at time 25 is:

$$AV_b = \int_a^b Pmt_t e^{\int_t^b r_s ds}dt$$

$$7{,}750 = \int_0^{25} x(6+t)e^{\int_t^{25}(6+s)^{-1}ds}dt$$

We begin by evaluating the integral in the exponent:

$$\int_t^{25}(6+s)^{-1}ds = \ln(6+s)\Big|_t^{25} = \ln(31) - \ln(6+t) = \ln\left(\frac{31}{6+t}\right)$$

The equation of value can now be used to solve for x:

$$7{,}750 = \int_0^{25} x(6+t)\left(\frac{31}{6+t}\right)dt$$

$$7{,}750 = \int_0^{25} 31x\,dt$$

$$7{,}750 = x(775-0)$$

$$x = \mathbf{10}$$

There are 3 common types of problems involving continuously varying payment streams:

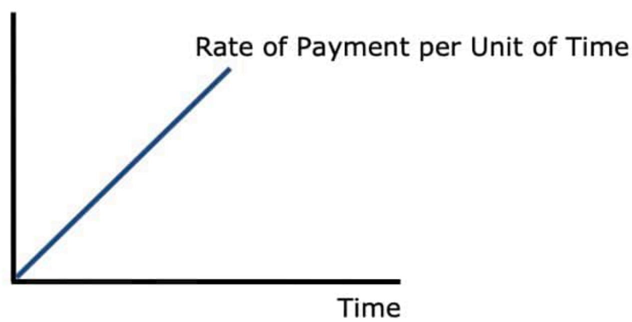
1. *The continuously compounded interest rate is constant. Example 9.01 is an example of this kind of question, and it is answered using integration by parts.*
2. *The varying payment is a multiple of r_t. Examples 9.02 and 9.03 are examples of this kind of question, and they are answered using substitution.*
3. *The varying payment is a multiple of a linear function of t, and r_t is the inverse of the linear function. Example 9.04 is an example of this kind of question, and it is fairly easy to answer because the linear functions cancel out.*

9.02 Annuities Increasing Continuously, Payable Continuously

In this section, we derive a formula for the present value of an annuity that pays at a rate of t per time unit at time t:

$$Pmt_t = t$$

The rate of payment increases with time:



We use $(\bar{I}\bar{a})_{\overline{n|}}$ to denote the present value of this annuity. If the force of interest is constant, then the present value is:

$$(\bar{I}\bar{a})_{\overline{n|}} = PV_0 = \int_0^n Pmt_t e^{-\int_0^t r\,ds} dt = \int_0^n te^{-tr} dt$$

We can use integration by parts:

$$\int u\,dv = uv - \int v\,du$$

$$u = t \qquad v = -\frac{1}{r}e^{-rt}$$

$$du = dt \qquad dv = e^{-rt}dt$$

The integral is:

$$\int te^{-rt}dt = -\frac{t}{r}e^{-rt} + \int \frac{1}{r}e^{-rt}dt = -\frac{t}{r}e^{-rt} - \frac{1}{r^2}e^{-rt}$$

Above, we used v to perform integration by parts, but below we switch to using v as the one-year discount factor, so $v = e^{-r}$. We have:

$$(\bar{I}\bar{a})_{\overline{n|}} = \int_0^n te^{-tr}dt = \left[-\frac{t}{r}e^{-rt} - \frac{1}{r^2}e^{-rt}\right]_0^n = \left[-\frac{n}{r}e^{-rn} - \frac{1}{r^2}e^{-rn}\right] - \left[0 - \frac{1}{r^2}\right]$$

$$= -\frac{nv^n}{r} - \frac{v^n}{r^2} + \frac{1}{r^2} = -\frac{nv^n}{r} + \frac{1-v^n}{r^2} = -\frac{nv^n}{r} + \frac{\bar{a}_{\overline{n|}}}{r} = \frac{\bar{a}_{\overline{n|}} - nv^n}{r}$$

The accumulated value of the annuity at time n is denoted by $(\bar{I}\bar{s})_{\overline{n|}}$, and its value can be found by accumulating for n units of time:

$$(\bar{I}\bar{s})_{\overline{n|}} = (1+i)^n(\bar{I}\bar{a})_{\overline{n|}} = (1+i)^n\frac{\bar{a}_{\overline{n|}} - nv^n}{r} = \frac{\bar{s}_{\overline{n|}} - n}{r}$$

9.02

Annuity Increasing Continuously, Payable Continuously

The present value and accumulated of an annuity that pays continuously at a rate of t per time unit at time t for n units of time are:

$$(\bar{I}\bar{a})_{\overline{n|}} = \frac{\bar{a}_{\overline{n|}} - nv^n}{r} \qquad (\bar{I}\bar{s})_{\overline{n|}} = \frac{\bar{s}_{\overline{n|}} - n}{r}$$

where:

$r =$ Force of interest

The annuity in the example below is the same as the one that appears in Example 9.01.

Example 9.05

A 10-year annuity makes continuous payments at an annual rate of $(100+10t)$ for $0 \le t \le 10$. The annual interest rate compounded monthly is 6%.

Calculate the present value of the annuity.

Solution

The continuously compounded interest rate is:

$$r = \ln\left(1+\frac{i^{(12)}}{12}\right)^{12} = \ln\left(1+\frac{0.06}{12}\right)^{12} = 0.05985$$

The present value is:

$$100\bar{a}_{\overline{10|}} + 10(\bar{I}\bar{a})_{\overline{10|}} = 100 \times \frac{1-v^{10}}{r} + 10 \times \frac{\bar{a}_{\overline{10|}} - 10v^{10}}{r}$$

$$= 100 \times \frac{1-1.005^{-10\times12}}{0.05985} + 10 \times \frac{\dfrac{1-1.005^{-120}}{0.05985} - 10 \times 1.005^{-10\times12}}{0.05985}$$

$$= 100 \times 7.5249 + 10 \times \frac{7.5249 - 5.4963}{0.05985}$$

$$= 752.4871 + 10 \times 33.8935 = \mathbf{1{,}091.42}$$

9.03 Questions

Question 9.01

A continuously increasing annuity with a term of n years has payments payable at an annual rate t at time t. The force of interest is equal to $1/n$.

Calculate the accumulated value of the annuity at time n.

A $n^2(e-1)$ B $n^2(e-2)$ C $n^2(e^{1/n}-1)$ D $n^2(2e^{1/n}-1)$ E $n^2(e^n-1)$

Question 9.02

Beginning at time 0, payments are made to an account at a rate of $(7k+tk)$.

Interest is credited at the following force of interest:

$$\delta_t = \frac{1}{7+t}, \text{ where } 0 \le t \le 12$$

At time 12, the accumulated value of the account is 22,344.

What is the value of k?

A 94 B 95 C 96 D 97 E 98

Question 9.03

A perpetuity makes continuous payments. At time t, the perpetuity pays at an annual rate of $3t$. Time is measured in years.

The annual effective interest rate is 5%.

Calculate the present value of the perpetuity.

A 1,141.24 B 1,229.76 C 1,259.25 D 1,260.25 E 1,291.24

Question 9.04

A 10-year annuity makes continuous payments. At time t, the annuity pays at an annual rate of $3t$. Time is measured in years.

The annual effective interest rate is 5%.

Calculate the accumulated value of the annuity at the end of 10 years.

A 132.59 B 150.84 C 158.51 D 177.69 E 197.18

Question 9.05

A 10-year annuity makes continuous payments. At time t, the annuity pays at an annual rate of $(30-3t)$. Time is measured in years.

The annual effective interest rate is 5%.

Calculate the present value of the annuity.

A 128.31 B 140.18 C 144.51 D 156.00 E 201.03

Question 9.06

Payments are made continuously at an annual rate of $(30t+20)$ for 8 years, beginning at time 0.

The force of interest at time t is $(0.03t+0.02)$, where t is measured in years.

Calculate the present value of the payments.

A 236.28 B 550.67 C 673.72 D 686.51 E 875.07

Question 9.07

From time 5 to time 10, payments are made continuously at an annual rate of $(3t^2 + 20t)$, where t is time measured in years. During the period that the continuous payments are made, the force of interest is $(0.006t^2 + 0.04t)$.

From time 0 to time 5, the force of interest is 5%.

Calculate the present value at time 0 of the payments.

A 374.30 B 376.57 C 378.75 D 480.61 E 483.53

Question 9.08

A perpetuity makes continuous payments, and the annual rate of payment at time t is:

$$\begin{cases} 1 & \text{for } 0 \le t < 20 \\ (1.05)^{t-20} & \text{for } t \ge 20 \end{cases}$$

The annual effective rate of interest is 8%.

Calculate the present value at time $t = 0$ of the perpetuity.

A 2.59 B 7.62 C 17.82 D 35.50 E 45.70

Question 9.09

An annuity makes continuous payments for 10 years. The payments increase for the first 5 years and decrease for the final 5 years. The annual rate of payment at time t is:

$$\begin{cases} 10t & \text{for } 0 \le t < 5 \\ 10(10 - t) & \text{for } 5 \le t \le 10 \end{cases}$$

The annual effective rate of interest is 5%.

Calculate the present value at time $t = 0$ of the annuity.

A 134.81 B 196.85 C 221.84 D 244.88 E 370.67

Question 9.10

You are given that the force of interest at time t is $0.001t^3$.

Calculate the present value of a four-year continuously increasing annuity that pays at a rate of $3t^3$ at time t.

A 186 B 198 C 937 D 995 E 1,418

Chapter 10: Geometric Progression Annuities

10.01 Geometric Progression Annuity Formulas

An annuity with payments that are a product of the preceding payment and a constant factor is known as a geometric progression annuity.

Consider an annuity that pays 1 at the beginning of the first unit of time, $(1+g)$ at the beginning of the second unit of time, $(1+g)^2$ at the beginning of the third year, and so on until the final payment of $(1+g)^{n-1}$ at the beginning of the n^{th} year.

We examine the geometric annuity-due before the geometric annuity-immediate in this section because we can obtain a convenient formula for the value of the annuity-due, which can then be adjusted to find the value of the corresponding annuity-immediate.

Payment	1	$(1+g)$	$(1+g)^2$		…	$(1+g)^{n-2}$	$(1+g)^{n-1}$	
Time	0	1	2	3	…	$(n-2)$	$n-1$	n

If g is positive, then the payments are increasing, and their rate of growth is g.

The present value of this annuity is:

$$PV_0 = 1 + (1+g)v + (1+g)^2v^2 + (1+g)^3v^3 + \cdots + (1+g)^{(n-1)}v^{n-1}$$

$$= 1 + \left(\frac{1+g}{1+i}\right) + \left(\frac{1+g}{1+i}\right)^2 + \left(\frac{1+g}{1+i}\right)^3 + \cdots + \left(\frac{1+g}{1+i}\right)^{(n-1)}$$

Let's define j as follows:

$$1 + j = \frac{1+i}{1+g} \qquad \Rightarrow \qquad j = \frac{i-g}{1+g}$$

The present value of the annuity can now be written as:

$$PV_0 = 1 + \left(\frac{1+g}{1+i}\right) + \left(\frac{1+g}{1+i}\right)^2 + \left(\frac{1+g}{1+i}\right)^3 + \cdots + \left(\frac{1+g}{1+i}\right)^{(n-1)}$$

$$= 1 + \left(\frac{1}{1+j}\right) + \left(\frac{1}{1+j}\right)^2 + \left(\frac{1}{1+j}\right)^3 + \cdots + \left(\frac{1}{1+j}\right)^{(n-1)}$$

$$= \ddot{a}_{\overline{n}|j}$$

The accumulated value of the annuity at time n is found by accumulating the present value for n years:

$$AV_n = (1+i)^n \ddot{a}_{\overline{n}|j}$$

Even though we can use j find the present value, we cannot use j to accumulate the present value. To accumulate the present value, we must use i.

Geometric Annuity-Due, Payable Once per Time Unit

10.01

The present value at time 0 and the accumulated value at time n of an annuity that pays 1 at the beginning of the first unit of time, $(1+g)$ at the beginning of the second unit of time, $(1+g)^2$ at the beginning of the third year, and so on until the final payment of $(1+g)^{n-1}$ at the beginning of the n^{th} year are:

$$PV_0 = \ddot{a}_{\overline{n}|j} \qquad\qquad AV_n = (1+i)^n \ddot{a}_{\overline{n}|j}$$

where:

$$1+j = \frac{1+i}{1+g}$$

If a geometric annuity makes each of its payments at the end of each unit of time, then it is called a geometric annuity-immediate.

Payment		1	$(1+g)$	$(1+g)^2$	...	$(1+g)^{n-3}$		$(1+g)^{n-1}$
Time	0	1	2	3	...	$(n-2)$	$n-1$	n

Since the payments are made one period later than the payments from the corresponding geometric annuity-due, the present value and the accumulated value of the payments can be obtained by dividing the present value and the accumulated value of the geometric annuity-due by $(1+i)$.

Geometric Annuity-Immediate, Payable Once per Time Unit

10.02

The present value at time 0 and the accumulated value at time n of an annuity that pays 1 at the end of the first unit of time, $(1+g)$ at the end of the second unit of time, $(1+g)^2$ at the end of the third year, and so on until the final payment of $(1+g)^{n-1}$ at the end of the n^{th} year are:

$$PV_0 = \frac{\ddot{a}_{\overline{n}|j}}{1+i} \qquad\qquad AV_n = (1+i)^{n-1} \ddot{a}_{\overline{n}|j}$$

where:

$$1+j = \frac{1+i}{1+g}$$

Example 10.01

An annual, 10-year geometric annuity-due makes a payment of \$100 now. Each subsequent payment is 7% greater than its preceding payment. The annual effective interest rate is 5%.

Calculate the present value of the annuity-due.

Solution

The value of j is:

$$j = \frac{i-g}{1+g} = \frac{0.05-0.07}{1.07} = -0.01869$$

The present value is:

$$PV_0 = 100\ddot{a}_{\overline{n}|j} = 100 \times \frac{1-(1-0.01869)^{-10}}{\frac{-0.01869}{1-0.01869}} = 100 \times 10.9022 = \mathbf{1{,}090.22}$$

Using the BA-II Plus, we have:

0.05 [−] 0.07 [=] [÷] 1.07 [×] 100 [=] [*I/Y*]
10 [*N*] 100 [*PMT*] 0 [*FV*]
[2nd] [BGN] [2nd] [SET] [2nd] [QUIT]
[CPT] [*PV*]
$PV = -1{,}090.22$
Solution = **1,090.22**

10.02 Geometric Progression Annuity as a Geometric Series

Problems involving geometric annuities can also be solved using the formula for a geometric series:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$
$$= \frac{\text{First term} - \text{Term that would come next}}{1 - \text{Ratio}}$$

Example 10.02 An annual, 10-year geometric annuity-due makes a payment of \$100 now. Each subsequent payment is 7% greater than its preceding payment. The annual effective interest rate is 5%.

Calculate the present value of the annuity-due.

Solution The present value is:

$$PV_0 = 100 + \frac{100 \times 1.07}{1.05} + \frac{100 \times 1.07^2}{1.05^2} + \cdots + \frac{100 \times 1.07^9}{1.05^9}$$
$$= \frac{100 - 100 \times \left(\frac{1.07}{1.05}\right)^{10}}{1 - \frac{1.07}{1.05}} = \mathbf{1,090.22}$$

In the example below, the geometric annuity makes payments monthly, and it increases monthly as well.

Example 10.03 A 5-year geometric annuity-immediate makes monthly payments. The first monthly payment is \$50, and each subsequent payment is 0.5% greater than its predecessor. The annual effective interest rate is 10%.

Calculate the accumulated value of the annuity at the end of 5 years.

Solution The monthly effective interest rate is:

$$\frac{i^{(12)}}{12} = 1.10^{\frac{1}{12}} - 1 = 0.007974$$

The value of j is

$$j = \frac{i-g}{1+g} = \frac{0.007974 - 0.005}{1.005} = 0.002959$$

The accumulated value at the end of 60 months is:

$$AV_{60} = 50 \times (1+i)^{n-1} \ddot{a}_{\overline{n|}j} = 50 \times 1.007974^{59} \times \frac{1 - 1.002959^{-60}}{\frac{0.002959}{1.002959}}$$
$$= 50 \times 1.007974^{59} \times 55.0632 = \mathbf{4,398.92}$$

Alternatively, we can find the accumulated value directly as the sum of a geometric series:

$$AV_{60} = 50(1.10)^{\frac{59}{12}} + 50(1.005)(1.10)^{\frac{58}{12}} + 50(1.005)^2(1.10)^{\frac{57}{12}} + \cdots + 50(1.005)^{59}$$

$$= 50 \times \frac{1.10^{\frac{59}{12}} - 1.005^{60} \times 1.10^{-\frac{1}{12}}}{1 - \frac{1.005}{1.10^{\frac{1}{12}}}} = 50 \times \frac{1.5978 - 1.3382}{0.002951} = 50 \times 87.9783$$

$$= \mathbf{4{,}398.92}$$

As long as the growth rate is less than the effective interest rate, then we can find the present value of a geometric increasing perpetuity. If the rate of growth of a perpetuity is greater than the effective interest rate, then the perpetuity's present value is infinite.

Example 10.04

An annual, geometric perpetuity-due makes a payment of $100 now. Each subsequent payment is 3% greater than its preceding payment. The annual effective interest rate is 10%.

Calculate the present value of the perpetuity-due.

Solution

The value of j is:

$$j = \frac{i-g}{1+g} = \frac{0.10-0.03}{1.03} = 0.06796$$

The present value of the perpetuity is:

$$PV_0 = 100 \times \ddot{a}_{\overline{n|}j} = 100 \times \ddot{a}_{\overline{\infty|}0.06796} = 100 \times \frac{1-0}{\frac{0.06796}{1.06796}} = 100 \times 15.7143 = \mathbf{1{,}571.43}$$

Alternatively, we can find the present value directly as the sum of an infinite geometric series:

$$PV_0 = 100 + \frac{100 \times 1.03}{1.10} + \frac{100 \times 1.07^2}{1.10^2} + \cdots = \frac{100-0}{1-\frac{1.03}{1.10}} = \mathbf{1{,}571.43}$$

Finding the value of the geometric series directly is often quicker than using the formulas in the preceding section.

10.03 Dividend Discount Model for Common Stock

A corporation is owned by the holders of its common stock. The owners of the shares of common stock are entitled to vote on how the corporation is run, and they are also entitled to dividends that are distributed from the profits of the corporation.

The value of a share of common stock can be estimated as the present value of the future dividends that will be paid to the owner of the stock. This method of finding the value of the stock is known as the dividend discount model.

10.03

Dividend Discount Model

The value of a share of stock is the present value of its future dividends:

$$PV_0 = \sum_{t>0} \frac{Div_t}{(1+i)^t}$$

where:

Div_t = Dividend payable at time t

i = Required rate of return for the stock

If the dividends are payable at the end of each unit of time and increase by a constant rate of growth g, then the present value is an infinite geometric series:

$$PV_0 = \frac{Div_1}{1+i} + \frac{Div_1 \times (1+g)}{(1+i)^2} + \frac{Div_1 \times (1+g)^2}{(1+i)^3} + \cdots$$

If the growth rate g is less than i, then the additions to the series tend to zero, and we have:

$$PV_0 = \frac{\text{First term} - \text{Term that would come next}}{1 - \text{Ratio}} = \frac{\dfrac{Div_1}{(1+i)} - 0}{1 - \dfrac{1+g}{1+i}} = \frac{Div_1}{1+i-(1+g)}$$

$$= \frac{Div_1}{i-g}$$

This version of the dividend discount model deals with dividend growth, so we refer to it as the dividend discount growth model. It is also known as the Gordon growth model, named after Myron J. Gordon, who published it in 1956.

10.04

Dividend Discount Growth Model

If the dividends of a stock are payable at the end of each unit of time and increase by a constant rate of growth equal to g, then the value of the stock is:

$$PV_0 = \frac{Div_1}{i-g} \qquad \text{where: } g < i$$

Example 10.05

A stock pays a dividend of \$10 in 1 year. Thereafter, the dividends grow at a rate of 2% per year forever. The required rate of return on the stock is an annual effective rate of 5%.

Calculate the value of the stock.

Solution

The value of the stock is:

$$PV_0 = \frac{Div_1}{i-g} = \frac{10}{0.05-0.02} = \mathbf{333.33}$$

In the Key Concept above, the payment frequency is the same as the interest conversion period of the required rate of return. If the required rate of return is provided as a rate with a different interest conversion period, then it may be necessary to convert the required rate into an equivalent interest rate with the same interest conversion period as the payment frequency.

Example 10.06

A stock pays a dividend of \$10 in 1 year. Thereafter, the dividends grow at a rate of 2% per year forever. The required rate of return on the stock is 5%, convertible twice per year.

Calculate the value of the stock.

Solution

The dividends are payable once per year, so we need to express the required rate of return as a rate that compounded once per year:

$$i = \left(1 + \frac{i^{(2)}}{2}\right)^2 - 1 = 1.025^2 - 1 = 0.050625$$

The value of the stock is:

$$PV_0 = \frac{Div_1}{i-g} = \frac{10}{0.050625-0.02} = \mathbf{326.53}$$

This section is just an introduction to the topic of stock valuation. We've ignored the difficulty of obtaining accurate estimates for the future dividends. In reality, dividends are not known in advance with certainty.

10.04 Questions

Question 10.01

An annuity-due makes 10 annual payments. The first payment of 100 is made now, and each subsequent payment is 12% greater than its preceding payment. The annual effective interest rate is 4%.

Calculate the accumulated value of the annuity at the end of 10 years.

A 1,427.66 B 1,655.75 C 1,962.34 D 2,113.29 E 2,275.85

Question 10.02

An annuity-immediate makes quarterly payments for 10 years. The first payment of 100 is made in 3 months, and each subsequent payment is 2% greater than its preceding payment. The annual effective interest rate is 10%.

Calculate the present value of the annuity.

A 3,614.88 B 3,698.0 C 3,702.05 D 3,787.18 E 3,791.32

Question 10.03

An annuity-immediate makes quarterly payments for 10 years. The first payment of 100 is made in 3 months, and each subsequent payment is 2% less than its preceding payment. The annual effective interest rate is 10%.

Calculate the future value of the annuity at the end of 10 years.

A 1,884.16 B 1,877.34 C 4,659.59 D 4,869.33 E 4,887.01

Question 10.04

A perpetuity-due makes annual payments that increase by 3% per year. The first payment is 36.

The force of interest is equal to 7%.

Calculate the present value of the perpetuity.

A 908.30 B 935.55 C 944.30 D 963.00 E 971.55

Question 10.05

Jason will receive a series of payments at the beginning of each year for 20 years. The first payment is 100. The subsequent 9 payments increase by 10% from the previous payment. After the 10^{th} payment, each payment decreases by 10% from the previous payment.

The annual effective interest rate is 4%.

Calculate the present value of these payments at the time the first payment is made.

A 2,078.99 B 2,118.01 C 2,164.98 D 2,193.24 E 2,516.16

Question 10.06

An employer must pay medical costs for an employee that was injured on a job site. Annual medical costs today are 4,000 for the kind of injury sustained by the employee, and the medical costs will increase by 8% per year. Fifteen payments will be made.

The medical costs are paid annually, and the first payment is to be made one year from today.

The annual effective interest rate is 6%.

Calculate the present value of the medical costs.

A 64,610.69 B 64,727.07 C 67,192.64 D 68,610.69 E 69,905.23

Question 10.07

A perpetuity-immediate pays 200 per year. Immediately after the tenth payment, the perpetuity is exchanged for a 20-year annuity-immediate that pays X one year after the exchange. Each of the subsequent 19 payments of the annuity-immediate is 4% greater than its predecessor.

The annual effective interest rate is 4%.

Calculate X.

A 240 B 250 C 260 D 270 E 280

Question 10.08

A perpetuity-immediate makes annual payments. The perpetuity-immediate makes level payments of 20 for the first 5 years. Beginning in year 6, each payment is K% greater than its preceding payment.

You are given:

- K is less than 8.
- Then annual effective interest rate is 8%.
- The present value of the perpetuity-immediate is 486.26.

Calculate the value of K.

A 4.4 B 4.5 C 4.6 D 4.7 E 4.8

Question 10.09

A loan is amortized over 4 years with monthly payments at an annual nominal interest rate of 6% compounded monthly. The first payment is 10,000, and it is paid one month after the origination of the loan. Each succeeding monthly payment is 1% lower than the payment that precedes it.

Calculate the outstanding balance of the loan after the 38^{th} payment is made.

A 62,895.95 B 63,531.26 C 63,848.92 D 64,172.99 E 64,493.86

Question 10.10

A loan is to be repaid with 18 annual payments:

The first payment of $3,000 is due one year from now.

The next 7 payments are each 4% larger than the preceding payment

For the 9^{th} through the 18^{th} payment, each payment will be 4% less than the preceding payment.

The annual effective interest rate of the loan is 6%.

Calculate the present value of the loan payments.

A 36,152.23 B 36,869.85 C 37,049.28 D 37,424.31 E 38,321.36

Question 10.11

A retiree will receive monthly payments for 15 years starting one month after retirement. For the first year, the amount of each monthly payment is 600. For each subsequent year, the monthly payments are 3% more than the monthly payments from the previous year.

Using an annual nominal interest rate of 9% compounded monthly, the present value of the payments upon retirement is X.

Calculate X.

A 61,886.42 B 63,009.99 C 69,870.94 D 72,655.98 E 74,263.70

Question 10.12

A man worked for 25 years before retiring. At the end of the first year of employment, he deposited 2,000 into a retirement account. At the end of each subsequent year of employment, he deposited g% more than the prior year. The man made a total of 25 deposits.

He will withdraw 7,395.44 at the beginning of the first year of retirement and will make annual withdrawals at the beginning of each subsequent year for a total of 25 withdrawals. Each of these subsequent withdrawals will be g% more than the prior year's withdrawal. The final withdrawal depletes the account.

The account earns a constant annual effective interest rate of i.

Calculate i.

A 4.96% B 5.16% C 5.25% D 5.37% E 5.60%

Question 10.13

An annuity-immediate pays one per year for the first two years, increasing by 8.16% every 2 years. The annuity is payable for 10 years.

Using an annual effective interest rate of 4%, determine an expression for the present value of this annuity.

A $(1+v)a_{\overline{5}|}$ B $(1+v)\ddot{a}_{\overline{5}|}$ C $2a_{\overline{5}|}$ D $5a_{\overline{2}|}$ E $\dfrac{a_{\overline{5}|}}{a_{\overline{1}|}}$

Question 10.14

An annuity-immediate pays one per year for the first 4 years, increasing by 8.16% every 4 years. The annuity is payable for 20 years.

Using an annual effective interest rate of 4%, determine an expression for the present value of this annuity.

A $(1+v^2)a_{\overline{10}|}$ B $(1+v^2)\ddot{a}_{\overline{10}|}$ C $4a_{\overline{10}|}$ D $10a_{\overline{4}|}$ E $\dfrac{a_{\overline{20}|}}{a_{\overline{2}|}}$

Question 10.15

A man worked for 25 years before retiring. At the end of the first year of employment, he deposited 2,000 into a retirement account. At the end of each subsequent year of employment, he deposited 2% more than the prior year. The man made a total of 25 deposits.

He will withdraw 7,395.44 at the beginning of the first year of retirement and will make annual withdrawals at the beginning of each subsequent year for a total of 25 withdrawals. Each of these subsequent withdrawals will be 2% more than the prior year's withdrawal. The final withdrawal depletes the account.

The account earns a constant annual effective interest rate.

Calculate the account balance after the final deposit and before the first withdrawal.

A 98,241 B 121,500 C 125,788 D 128,304 E 132,832

Question 10.16

The price of a stock is 175 per share assuming an annual effective interest rate of *i*.

The stock pays a dividend of 7 in one year, and thereafter annual dividends increase by 2% per year. The annual dividends are paid forever.

Calculate *i*.

A 0.02 B 0.03 C 0.04 D 0.05 E 0.06

Question 10.17

A company enters into a transaction that requires it to deposit 1,000 now and 130 per year into perpetuity. The first payment of 130 is made at the end of one year.

In return, the company receives a perpetuity that makes annual payments. The first payment is 100, and it is received at the end of 1 year. Each subsequent payment is 3% greater than the payment that precedes it.

Calculate the yield for this transaction.

A 2.08% B 3.10% C 3.90% D 4.33% E 6.24%

Question 10.18

Jim purchases a perpetuity-due for 316,965.95, based on an annual effective interest rate of *i*. The perpetuity-due provides a geometric series of quarterly payments. The first quarterly payment is 1,700 and the second quarterly payment is 1,704.25. Each subsequent quarterly payment increases by this same ratio.

Calculate *i*.

A 3.18% B 3.20% C 5.00% D 5.41% E 7.91%

Question 10.19

A share of common stock will pay dividends of 2 at the end of each of the next 10 years and 4 at the end of each of the subsequent 10 years. Thereafter the dividends increase by 1% per year.

The annual effective interest rate is 5%.

Calculate the price of the stock.

A 72.09 B 73.98 C 72.47 D 76.09 E 84.40

Question 10.20

Stock M and Stock N are both valued using an annual effective interest rate of 7%. Both stocks pay annual dividends forever, beginning in one year.

Stock M's dividends increase at an annual growth rate of g, and Stock N's dividends have an annual growth rate of $-g$.

The value of Stock M is 3 times the value of Stock N, and the next dividend paid by Stock M is one third of the next dividend paid by Stock N.

Calculate g.

A 0.035	B 0.056	C 0.062	D 0.070	E 0.078

Chapter 11: Loans

"Neither a borrower nor a lender be…"

-Polonius, giving advice in Hamlet, by William Shakespeare

While this may be good advice for friends and family, loans are a common way of providing funds immediately in exchange for future payments. When a loan is made, a lender provides funds to the borrower. The borrower agrees to return the funds plus interest to the lender.

When a loan is gradually paid down over time, the loan is said to be amortized over time. An amortized loan has more than one principal repayment date.

11.01 Amortized Loans, General Case

Consider a loan of L_0 that is to be repaid with a series of payments over n units of time. The term of the loan is n units of time. Let's denote a payment at time t as Pmt_t. The balance of the loan at the outset is equal to the present value of the payments to be made in the future:

$$L_0 = PV_0(\text{Payments})$$

Each payment can be broken down into its components of interest and principal:

$$Int_t = \text{Interest portion of the time-}t\text{ payment}$$

$$Prn_t = \text{Principal portion of the time-}t\text{ payment}$$

The sum of the two components is equal to the payment amount:

$$Pmt_t = Int_t + Prn_t$$

The interest component is the product of the loan balance remaining after the previous payment and the periodic interest rate appropriate for the length of the period. The principal component is the portion of the payment that is not allocated to interest:

$$Int_t = (\text{Preceding Loan Balance}) \times (\text{Periodic interest rate})$$

$$Prn_t = Pmt_t - Int_t$$

The loan balance at time t is denoted by L_t, and it is equal to the previous loan balance minus the principal payment made at time t:

$$L_t = (\text{Previous Loan Balance}) - Prn_t$$

Example 11.01

A loan is to be repaid with payments of $250 in 1 year, $300 in 2 years, $100 in 3 years, and $490.35 in 4 years. The annual effective interest rate is 5%.

a. What is the initial amount of the loan?

b. What is the interest portion of the first payment?

c. What is the principal portion of the first payment?

d. What is the loan balance after the first payment?

Solution

a. The initial amount of the loan is:

$$L_0 = PV_0(\text{Payments}) = \frac{250}{1.05} + \frac{300}{1.05^2} + \frac{100}{1.05^3} + \frac{490.35}{1.05^4} = \mathbf{1{,}000}$$

Alternatively, using the BA-II Plus:

[CF] ↓ 250 [ENTER] ↓↓ 300 [ENTER] ↓↓ 100 [ENTER]

↓↓ 490.35 [ENTER]

[2nd] [QUIT]

[NPV] 5 [ENTER] ↓ [CPT]

Result = **1,000**

b. The interest portion of the first payment is:

$$Int_1 = 1{,}000 \times 0.05 = \mathbf{50}$$

c. The principal portion of the first payment is:

$$Prn_1 = Pmt_1 - Int_1 = 250 - 50 = \mathbf{200}$$

d. The loan balance after the first payment is:

$$L_1 = L_0 - Prn_1 = 1{,}000 - 200 = \mathbf{800}$$

In part a above, we used the net present value function of the BA-II Plus calculator to find the present value of a stream of cash flows. We will see this function again in Chapter 12.

L_t can be found by subtracting the principal payments that occur on or before time t from the initial loan balance:

$$L_t = L_0 - \sum_{s \le t} Prn_s$$

There are two methods for finding L_t that do not require knowledge of the principal payments occurring on or before time t. The initial value of the loan is equal to the present value of the payments. Another way to say this is that the time-0 current value of the loan is equal to the time-0 current value of the payments:

$$L_0 = PV_0(\text{Payments})$$
$$CV_0(\text{Loan}) = CV_0(\text{Payments})$$

When the current values of the two sets of payments are equal at one point in time, they are equal at all points in time. Let's consider the current values at time t:

$$CV_t(\text{Loan}) = CV_t(\text{Payments})$$
$$(1+i)^t L_0 = AV_t\begin{pmatrix}\text{Payments made on}\\ \text{or before time } t\end{pmatrix} + PV_t\begin{pmatrix}\text{Payments made}\\ \text{after time } t\end{pmatrix}$$

We can subtract the accumulated value of the payments made on or before time t from both sides:

$$(1+i)^t L_0 - AV_t\begin{pmatrix}\text{Payments made on}\\ \text{or before time } t\end{pmatrix} = PV_t\begin{pmatrix}\text{Payments made}\\ \text{after time } t\end{pmatrix}$$

The payments are made solely to pay off the loan, and this implies that at any time the outstanding loan balance is equal to the present value of the remaining payments. Therefore, the right side of the equation above is equal to the loan balance at time t. Since the left side is equal to the right side, the left side of the equation is also equal to the loan balance at time t. The left side represents the **retrospective method**, and it finds the loan balance at time t as the accumulated value of the loan minus the accumulated value of the payments made on or before time t. The right side represents the **prospective method**, and it finds the loan balance at time t as the present value of the remaining payments to be made after time t:

Retrospective: $L_t = (1+i)^t L_0 - AV_t(\text{Payments made on or before time } t)$

Prospective: $L_t = PV_t(\text{Payments made after time } t)$

Example 11.02 A loan of \$1,000 is to be repaid with payments of \$250 in 1 year, \$300 in 2 years, \$100 in 3 years, and \$490.35 in 4 years. The annual effective interest rate is 5%.

a. Using the retrospective method, what is the loan balance at the end of 2 years?

b. Using the prospective method, what is the loan balance at the end of 2 years?

Solution a. Using the retrospective method, the loan balance at the end of 2 years is:

$$(1+i)^2 L_0 - AV_2\begin{pmatrix}\text{Payments made on}\\ \text{or before time 2}\end{pmatrix} = 1.05^2 \times 1{,}000 - [250\text{x}1.05 + 300]$$

$$= \mathbf{540}$$

b. Using the prospective method, we obtain the same loan balance at the end of 2 years:

$$PV_2\begin{pmatrix}\text{Payments made}\\ \text{after time 2}\end{pmatrix} = \frac{100}{1.05} + \frac{490.35}{1.05^2} = \mathbf{540}$$

11.01

Amortized Loans, General Case

The present value of an amortized loan is equal to the present value of the loan payments:

$$L_0 = PV(\text{Payments})$$

The interest and principal components of the time t payment are:

$$Int_t = (\text{Preceding Loan Balance}) \times (\text{Periodic interest rate})$$
$$Prn_t = Pmt_t - Int_t$$

The loan balance at time t can be found using the retrospective method or the prospective method:

Retrospective: $L_t = (1+i)^t L_0 - AV_t(\text{Payments made on or before time } t)$

Prospective: $L_t = PV_t(\text{Payments made after time } t)$

where:

Int_t = Interest portion of the time-t payment

Prn_t = Principal portion of the time-t payment

L_t = Loan balance after the time-t payment

Using the retrospective formula, we can obtain a formula for the loan balance at time t in terms of a prior loan balance at time k:

$$L_k = (1+i)^k L_0 - AV_k\begin{pmatrix}\text{Payments made on}\\ \text{or before time } k\end{pmatrix}$$

$$(1+i)^{t-k} L_k = (1+i)^t L_0 - AV_t\begin{pmatrix}\text{Payments made on}\\ \text{or before time } k\end{pmatrix}$$

$$(1+i)^{t-k} L_k - AV_t\begin{pmatrix}\text{Payments made after}\\ k \text{ and on or before } t\end{pmatrix}$$
$$= (1+i)^t L_0 - AV_t\begin{pmatrix}\text{Payments made on}\\ \text{or before time } k\end{pmatrix} - AV_t\begin{pmatrix}\text{Payments made after}\\ k \text{ and on or before } t\end{pmatrix}$$

$$(1+i)^{t-k} L_k - AV_t\begin{pmatrix}\text{Payments made after}\\ k \text{ and on or before } t\end{pmatrix} = (1+i)^t L_0 - AV_t\begin{pmatrix}\text{Payments made on}\\ \text{or before time } t\end{pmatrix}$$

$$(1+i)^{t-k} L_k - AV_t\begin{pmatrix}\text{Payments made after}\\ k \text{ and on or before } t\end{pmatrix} = L_t$$

Rearranging the final line above, we see that the loan balance at time t is the prior loan balance at time k accumulated to time t minus the accumulated value of the loan payments made after time k:

$$L_t = (1+i)^{t-k} L_k - AV_t\,(\text{Payments made after } k \text{ and on or before } t)$$

If we replace k with 0, then we have the retrospective formula:

$$\text{Retrospective:} \quad L_t = (1+i)^t L_0 - AV_t(\text{Payments made on or before time } t)$$

Example 11.03

A loan is to be repaid with payments of $\$X$ in 1 year, $\$Y$ in 2 years, \$100 in 3 years, and \$490.35 in 4 years. The annual effective interest rate is 5%. You are given that the loan balance at the end of 2 years is \$540.

a. Using the retrospective method and the loan balance at the end of 2 years, what is the loan balance at the end of 3 years?

b. Using the prospective method, what is the loan balance at the end of 3 years?

Solution

a. Using the retrospective method and the loan balance at the end of 2 years, the loan balance at the end of 3 years is:

$$L_3 = (1+i)^{3-2} L_2 - AV_3\begin{pmatrix}\text{Payments made after} \\ \text{2 and on or before 3}\end{pmatrix} = 1.05 \times 540 - 100 = \mathbf{467}$$

b. Using the prospective method, we obtain the same loan balance at the end of 3 years:

$$PV_3\begin{pmatrix}\text{Payments made} \\ \text{after time 3}\end{pmatrix} = \frac{490.35}{1.05} = \mathbf{467}$$

11.02 Level Payment Amortized Loans

A loan of L_0 can be repaid in n equal payments, each of which occurs at the end of each unit of time. The amount of each payment is denoted by Pmt, and this amount can be obtained by solving an equation of value that sets the value of the loan equal to the present value of the payments:

$$L_0 = Pmt \times a_{\overline{n|}}$$

$$Pmt = \frac{L_0}{a_{\overline{n|}}}$$

Each payment can be broken down into its components of interest and principal:

$$Pmt = Int_t + Prn_t$$

where: Int_t = Interest portion of the t^{th} payment

Prn_t = Principal portion of the t^{th} payment

The remaining balance of the loan at time t is L_t. The interest component is the previous balance times the periodic interest rate, and the principal component is the portion of the level payment that is not allocated to interest:

$$Int_t = L_{t-1} \times i$$

$$Prn_t = Pmt - Int_t$$

The loan balance at time t is the previous loan balance minus the principal payment made at time t:

$$L_t = L_{t-1} - Prn_t$$

We can use these relationships to create an amortization schedule that specifies the interest and the principal portions of each payment.

Example 11.04

A loan of \$10,000 is to be repaid with level annual payments over 5 years. The annual effective interest rate is 5%.

a. What is the amount of each annual payment?

b. Create an amortization table showing the principal and interest components of each payment. Also show the outstanding principal at the end of each year.

Solution

a. The payment amount is:

$$Pmt = \frac{L_0}{a_{\overline{n}|}} = \frac{10,000}{a_{\overline{5}|}} = \frac{10,000}{\frac{1-1.05^{-5}}{0.05}} = \frac{10,000}{4.3295} = \mathbf{2,309.75}$$

We can also find this value using the BA II Plus calculator:

5 [*N*] 5 [*I/Y*] 10,000 [*PV*] [CPT] [*PMT*]

PMT = –2,309.75 Answer = **2,309.75**

b. The interest and principal portions of the first payment and the resulting outstanding principal is obtained for the first year below:

$$Int_1 = L_0 \times i = 10,000 \times 0.05 = 500$$
$$Prn_1 = Pmt - Int_1 = 2,309.75 - 500 = 1,809.75$$
$$L_1 = L_0 - Prn_1 = 10,000 - 1,809.75 = 8,190.25$$

For the second year, the values are:

$$Int_2 = L_1 \times i = 8,190.25 \times 0.05 = 409.51$$
$$Prn_2 = Pmt - Int_2 = 2,309.75 - 409.51 = 1,900.24$$
$$L_2 = L_1 - Prn_2 = 8,190.25 - 1,900.24 = 6,290.02$$

For the third year, the values are:

$$Int_3 = L_2 \times i = 6,290.02 \times 0.05 = 314.50$$
$$Prn_3 = Pmt - Int_3 = 2,309.75 - 314.50 = 1,995.25$$
$$L_3 = L_2 - Prn_3 = 6,290.02 - 1,995.25 = 4,294.77$$

For the fourth year, the values are:

$$Int_4 = L_3 \times i = 4,294.77 \times 0.05 = 214.74$$
$$Prn_4 = Pmt - Int_4 = 2,309.75 - 214.74 = 2,095.01$$
$$L_4 = L_3 - Prn_4 = 4,294.77 - 2,095.01 = 2,199.76$$

For the fifth year, the values are:

$$Int_5 = L_4 \times i = 2,199.76 \times 0.05 = 109.99$$
$$Prn_5 = Pmt - Int_4 = 2,309.75 - 109.99 = 2,199.76$$
$$L_5 = L_4 - Prn_5 = 2,199.76 - 2,199.76 = 0$$

The values calculated above are used to create the amortization table below:

Year	Payment	Principal Payment	Interest Payment	Loan Balance
0				10,000.00
1	2,309.75	1,809.75	500.00	8,190.25
2	2,309.75	1,900.24	409.51	6,290.02
3	2,309.75	1,995.25	314.50	4,294.77
4	2,309.75	2,095.01	214.74	2,199.76
5	2,309.75	2,199.76	109.99	0.00

We can use the BA II Plus calculator to find the values above:

5 [*N*] 5 [*I/Y*] 10,000 [*PV*] [CPT] [*PMT*]

PMT = –2,309.75 Annual Payment = **2,309.75**

We can use the calculator's amortization worksheet to find the interest and principal portions of each payment.

For example, to obtain the interest and principal portions of the third payment we use the calculator as follows:

[2nd] [AMORT] 3 [ENTER] ↓ 3 [ENTER] ↓

By continuing to hit the down arrow key we observe:

BAL = 4,294.77

PRN = –1,995.25

INT = –314.50

The settings for P1 and P2 tell the calculator to determine the principal and interest included in payments P1 through P2. When we want to find the principal and interest components of a single payment, we set P1 equal to P2.

In the preceding section, we saw that the expression below provides two ways to find the outstanding balance of a loan at time t:

$$(1+i)^t L_0 - AV_t\begin{pmatrix}\text{Payments made on}\\ \text{or before time } t\end{pmatrix} = PV_t\begin{pmatrix}\text{Payments made}\\ \text{after time } t\end{pmatrix}$$

When the payments are level, this can be written as:

$$L_0(1+i)^t - Pmt \times s_{\overline{t|}} = Pmt \times a_{\overline{n-t|}}$$

The right side of the equation above is the present value of the payments occurring after time t, which is the value of the loan at time t. The use of the present value of the future payments to determine the value of the loan is known as the prospective method:

$$\text{Prospective:}\quad L_t = Pmt \times a_{\overline{n-t|}}$$

Alternatively, we can use the left side of the equation to find L_t as the accumulated value of the loan minus the accumulated value of the payments made up to and including time t. This is known as the retrospective method:

$$\text{Retrospective:}\quad L_t = L_0(1+i)^t - Pmt \times s_{\overline{t|}}$$

We can also find the loan balance in terms of a prior loan balance at time k:

$$L_t = L_k(1+i)^{t-k} - AV_t\begin{pmatrix}\text{Payments made after}\\ k \text{ and on or before } t\end{pmatrix} = L_k(1+i)^{t-k} - Pmt \times s_{\overline{t-k|}}$$

Example 11.05

A loan is repaid with level annual payments of $2,309.75. The annual effective interest rate is 5%. At the end of 2 years, the outstanding balance of the loan is $6,290.02.

What is the outstanding balance at the end of 4 years?

Solution

The outstanding balance is:

$$L_4 = (1+i)^{4-2}L_2 - Pmt \times s_{\overline{4-2|}} = 1.05^2 \times 6{,}290.02 - 2{,}309.75 s_{\overline{2|}}$$
$$= 6{,}934.75 - 2{,}309.75 \times 2.05 = \mathbf{2{,}199.76}$$

We can also find this value using the BA II Plus calculator:

2 [*N*] 5 [*I/Y*] –6,290.02 [*PV*] 2,309.75 [*PMT*] [CPT] [*FV*]

FV = **2,199.76**

We can find the percentage of the t^{th} payment that consists of principal:

$$\frac{Prn_t}{Pmt} = \frac{Pmt - Int_t}{Pmt} = \frac{Pmt - L_{t-1} \times i}{Pmt} = \frac{Pmt - Pmt \times a_{\overline{n-(t-1)|}} \times i}{Pmt}$$
$$= 1 - a_{\overline{n-(t-1)|}} \times i = 1 - (1 - v^{n-t+1}) = v^{n-t+1}$$

The percentage of the t^{th} payment that consists of interest is the complement of the percentage that consists of principal:

$$\frac{Int_t}{Pmt} = 1 - v^{n-t+1}$$

The portion of the payment that consists of principal increases at the rate of interest:

$$\frac{Prn_{t+k}}{Pmt} = v^{n-t-k+1} \quad \& \quad \frac{Prn_t}{Pmt} = v^{n-t+1}$$

$$\Rightarrow \quad \frac{Prn_{t+k}}{Prn_t} = \frac{v^{n-t-k+1}}{v^{n-t+1}} = v^{-k} = (1+i)^k$$

11.02

Loans Repaid with Level Payments

If a loan of L_0 is repaid with n installments, each of which is equal to Pmt, then:

$$Pmt = \frac{L_0}{a_{\overline{n}|}}$$

$$Int_t = L_{t-1} \times i = (1 - v^{n-t+1})Pmt$$

$$Prn_t = Pmt - Int_t = v^{n-t+1} \times Pmt$$

$$\frac{Prn_{t+k}}{Prn_t} = (1+i)^k$$

$$L_t = L_0(1+i)^t - Pmt \times s_{\overline{t}|} = Pmt \times a_{\overline{n-t}|}$$

$$L_t = L_k(1+i)^{t-k} - Pmt \times s_{\overline{t-k}|} \quad 0 \le k \le t$$

Example 11.06

A loan is repaid with 5 level annual payments of $2,309.75. The annual effective interest rate is 5%. Determine the interest and principal components of the 3rd payment.

Solution

The percentage of the 3rd payment that consists of principal is:

$$\frac{Prn_3}{Pmt} = v^{5-3+1} = 1.05^{-3} = 0.8638$$

The principal and interest components are:

$$Prn_3 = 0.8638 \times 2,309.75 = \mathbf{1,995.25}$$

$$Int_3 = (1 - 0.8638) \times 2,309.75 = \mathbf{314.50}$$

11.03 Balloon and Drop Payments

When a loan is paid off by a sequence of level payments followed by a final payment that is not equal to the level payments, the final payment is either a **balloon payment** or a **drop payment**. If the final payment is greater than the level payment amount, then the final payment is a balloon payment. If the final payment is less than the level payment amount, then the final payment is a drop payment:

$$FinalPmt > Pmt \quad \Rightarrow \quad FinalPmt = \text{Balloon Payment}$$

$$FinalPmt < Pmt \quad \Rightarrow \quad FinalPmt = \text{Drop Payment}$$

where: $Pmt = \text{Level Payment Amount}$

The final payment is equal to the loan balance just before the final payment is made. If the final payment is made at the end of n payment periods, then:

$$FinalPmt = L_{n-1} \times (1+i)$$

It is common for a balloon payment loan to have level payments that are calculated based on a term that is longer than the time until the loan matures. If s is the term used to calculate the level payments, then the level payment amount for a balloon payment loan is:

$$Pmt = \frac{L_0}{a_{\overline{s}|}} \qquad \text{where: } n < s$$

Example 11.07 A loan of $10,000 is to be repaid with level annual payments based on a 20-year amortization schedule. These level payments will be made for 9 years, and then the loan will be repaid at the end of 10 years with a final balloon payment. The annual effective interest rate is 5%.

Calculate the amount of the balloon payment.

Solution The amount of each level payment is:

$$Pmt = \frac{L_0}{a_{\overline{s}|}} = \frac{10{,}000}{a_{\overline{20}|}} = \frac{10{,}000}{\frac{1-1.05^{-20}}{0.05}} = 802.4259$$

The balloon payment is:

$$FinalPmt = L_{n-1} \times (1+i) = \left[10{,}000 \times (1.05)^9 - 802.43 s_{\overline{9}|}\right](1.05)$$
$$= \left[15{,}513.28 - 802.43 \times 11.0266\right](1.05) = \mathbf{6{,}998.55}$$

We can also find this value using the BA II Plus calculator:

20 [*N*] 5 [*I/Y*] 10,000 [+/-] [*PV*] [CPT] [*PMT*]

9 [*N*] [CPT] [*FV*] × 1.05 [=]

Answer is **6,998.55**.

When the level payment amount does not pay off a loan in an integer number of years, we can use a larger final payment to pay off the loan. This larger payment is a balloon payment.

Example 11.08 A loan of $10,000 is to be repaid with level annual payments of $1,000 and a final balloon payment. The annual effective interest rate is 5%.

Calculate the amount of the balloon payment.

Solution We use the BA II Plus calculator to find the term of the loan:

5 [*I/Y*] 10,000 [+/-] [*PV*] 1,000 [*PMT*] [CPT] [*N*]

Result is 14.21

Since N is not an integer, let's find the balance at the end of 13 years and then accumulate it for one more year to find the balloon payment made at the end of the 14th year:

13 [*N*] [CPT] [*FV*] × 1.05 [=]

Result is **1,200.68**.

Another way to handle a non-integer number of payments is to make an extra, smaller payment. This smaller payment is a drop payment.

Example 11.09 A loan of $10,000 is to be repaid with level annual payments of $1,000 and a final drop payment. The annual effective interest rate is 5%.

Calculate the amount of the drop payment.

Solution We use the BA II Plus calculator to find the term of the loan:

5 [*I/Y*] 10,000 [+/-] [*PV*] 1,000 [*PMT*] [CPT] [*N*]

Result is 14.21

Since N is not an integer, let's find the balance at the end of 14 years and then accumulate it for one more year to find the drop payment made at the end of the 15th year:

14 [N] [CPT] [FV] $\times$ 1.05 =

Result is **210.72**.

11.04 Sinking Fund Loans

From the perspective of the lender, a **sinking fund loan** is a loan with a balloon payment and an interest rate of i. From the perspective of the borrower, a sinking fund loan has two sets of payments:

1. Loan payments are applied directly to the loan. They are used to pay the interest on the loan, and if the loan payments exceed the interest, then the excess is used to pay the loan down.
2. Sinking fund payments are used to build up a separate fund, which we call a sinking fund. The sinking fund accumulates at an effective interest rate of j per unit of time. Upon the maturity of the loan, the accumulated value of the sinking fund is paid to the lender to pay off the loan. If the sinking fund payment is level, then it is denoted by SFP.

Upon maturity, the accumulated value of the loan minus the accumulated value of the loan payments is equal to the accumulated value of the sinking fund. Although the interest rate on the loan is i, the sinking fund accumulates at the sinking fund interest rate of j:

$$(1+i)^n L_0 - AV_n(\text{Loan payments, } i) = AV_n(\text{Sinking fund payments, } j)$$

The balance of the loan, L_t, and the balance of the sinking fund, SFB_t, at time t are shown below:

$$L_t = (1+i)^t L_0 - AV_t(\text{Loan payments, } i)$$
$$SFB_t = AV_t(\text{Sinking fund payments, } j)$$

The net balance of the loan, NL_t, at time t is the loan balance minus the sinking fund balance:

$$\begin{aligned} NL_t &= L_t - SFB_t \\ &= (1+i)^t L_0 - AV_t(\text{Loan payments, } i) - AV_t(\text{Sinking fund payments, } j) \end{aligned}$$

The calculation above is a retrospective calculation. There is no convenient prospective formula because the sinking fund balance at time t depends on past payments into the fund, not future payments.

Most sinking fund loans have the following two characteristics:

- *The loan payment is equal to the interest on the loan.*
- *The sinking fund payments are level.*

The Key Concept below is based on these two assumptions.

When the loan payment is equal to the interest on the loan, the loan balance at time t is equal to the initial loan balance:

$$\begin{aligned} L_t &= (1+i)^t L_0 - iL_0 \times s_{\overline{t|}i} = (1+i)^t L_0 - iL_0 \times \frac{(1+i)^t - 1}{i} \\ &= (1+i)^t L_0 - L_0 \times \left[(1+i)^t - 1\right] = L_0 \end{aligned}$$

The loan balance at time n is therefore L_0, and if the sinking fund payments are level, then the amount of each sinking fund payment is set so that the sinking fund payments accumulate to the loan amount:

$$SFP \times s_{\overline{n|}j} = L_0$$

Sinking Fund Loans

11.03

An n-year loan is to be repaid at time n with the balance of a sinking fund such that:

- The sinking fund payments are level and accumulate to the initial value of the loan at an interest rate of j.
- The periodic loan payments are equal to the interest on the loan.

The level sinking fund payment is:

$$SFP = \frac{L_0}{s_{\overline{n}|j}}$$

At time t, where $t \le n$, the outstanding balance of the loan, the balance of the sinking fund and the net amount of the loan are:

$$L_t = L_0$$
$$SFB_t = SFP \times s_{\overline{t}|j}$$
$$NL_t = L_0 - SFP \times s_{\overline{t}|j}$$

Example 11.10

A loan of $40,000 is to be repaid with annual payments using the sinking fund method over a period of 20 years. The annual effective interest rate on the loan is 6%, and the annual effective interest rate on the sinking fund is 4%. The annual payments made on the loan are equal to the interest due on the loan. The annual sinking fund payments accumulate for 20 years, at which time the loan is paid off.

a. Calculate the amount of each level sinking fund payment.
b. Calculate the loan balance at the end of 10 years.
c. Calculate the sinking fund balance at the end of 10 years.
d. Calculate the net amount of the loan at the end of 10 years.

Solution

a. Since the loan payment is equal to the interest payment, the value of the loan at the end of 20 years is the initial loan amount of $40,000. The sinking fund payment is:

$$SFP = \frac{40{,}000}{s_{\overline{20}|0.04}} = \frac{40{,}000}{\frac{1.04^{20}-1}{0.04}} = \frac{40{,}000}{29.7781} = \mathbf{1{,}343.27}$$

We can also use the BA II Plus calculator to find the sinking fund payment:

20 [*N*] 4 [*I/Y*] 40,000 [*FV*] [CPT] [*PMT*]

Result is −1,343.27 Answer is **1,343.27**

b. Since the loan payment is equal to the interest payment, the value of the loan remains constant at **$40,000**.

c. The sinking fund balance at the end of 10 years is:

$$1{,}343.27 s_{\overline{10}|0.04} = 1{,}343.27 \times \frac{1.04^{10}-1}{0.04} = 1{,}343.27 \times 12.0061$$
$$= \mathbf{16{,}127.44}$$

We can also use the BA II Plus calculator to find the accumulated value of the sinking fund. Below, we assume that the key strokes in part (a) have been performed, so the *PMT* amount is already stored as 1,343.27:

10 [*N*] [CPT] [*FV*]

Result is **16,127.44**

d. The net amount of the loan is the accumulated value of the loan minus the sinking fund balance:

$$40{,}000 - 16{,}127.44 = \mathbf{23{,}872.56}$$

If the sinking fund interest rate is equal to i, then the combination of the loan payment and the sinking fund payment is equal to the level payment that would be made under the regular amortization method:

$$L_0 \times i + SFP = L_0 \times i + \frac{L_0}{s_{\overline{n}|i}} = L_0\left[i + \frac{1}{s_{\overline{n}|i}}\right] = L_0\left[i + \frac{i}{(1+i)^n - 1}\right]$$

$$= L_0\left[\frac{i\left((1+i)^n - 1\right)}{(1+i)^n - 1} + \frac{i}{(1+i)^n - 1}\right] = L_0\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] = L_0\left[\frac{i}{1 - v^n}\right] = \frac{L_0}{a_{\overline{n}|i}} = Pmt$$

11.05 Questions

Question 11.01

A loan is being repaid with 45 annual payments of 400. Immediately after the 15^{th} payment, the borrower pays an extra 2,000. The balance is then repaid with level payments of X over the next 25 years.

The annual effective interest rate is 7%.

Calculate X.

A 254 B 291 C 327 D 364 E 400

Question 11.02

A borrower takes out a loan of 700. The annual interest rate is 8% convertible quarterly. The borrower makes payments of 14 at the end of each quarter.

Calculate the amount of principal paid in the fifth payment.

A 0 B 4 C 7 D 11 E 14

Question 11.03

A loan is repaid with level annual payments at the end of each year. The annual effective interest rate is 6%.

The 5^{th} payment consists of 73.09 of principal and 426.91 of interest.

Calculate the interest portion of the 17^{th} payment.

A 147 B 199 C 250 D 302 E 353

Question 11.04

Cara borrows a total of 12,000 for 5 years from 3 different lenders. The interest rate for all 3 loans is 8% compounded semiannually.

The first loan of 4,000 has the interest accumulated for 5 years and is repaid with a lump sum at the end of 5 years.

Cara pays the interest on the second loan of 4,000 every six months and repays the principal at the end of 5 years.

The third loan of 4,000 is repaid with 10 level payments that are made at the end of each 6-month period.

Calculate the total interest paid by Cara over the 5-year interval.

A 932 B 1,600 C 1,920 D 3,186 E 4,453

Question 11.05

A loan of 10,000 is repaid with 10 annual payments that are made at the end of each year. The annual effective interest rate is 10%.

Each of the first 5 payments equals 200% of the interest that is due. The second 5 payments are level payments, and the amount of each level payment is X.

Calculate X.

A 1,500 B 1,558 C 1,614 D 1,671 E 1,728

Question 11.06

Wally repays a loan with payments of 1 at the end of each year for n years. The amount of principal repaid in year t plus the amount of interest repaid in year $(t + 1)$ is equal to X.

The annual effective interest rate is i.

Which of the following expressions is equal to X?

A $\quad 1+\dfrac{v^{n-t}}{d}$

B $\quad 1-\dfrac{v^{n-t}}{d}$

C $\quad 1+\dfrac{v^{n-t}}{i}$

D $\quad 1+v^{n-t}d$

E $\quad 1-v^{n-t}d$

Question 11.07

A loan of X is to be repaid with equal payments at the end of each year for 5 years. The outstanding loan balance at the end of the fourth year is 911.74.

The annual effective interest rate of the loan is 7%.

Calculate the principal repaid with the first payment.

A 0 B 400 C 696 D 912 E 976

Question 11.08

A loan of X is to be repaid with equal payments at the end of each year for n years. The principal repaid with the sixth payment is 1,015.13.

The annual effective interest rate of the loan is 7%.

Calculate the principal repaid with the first payment.

A 652 B 676 C 700 D 724 E 748

Question 11.09

A borrower takes out a loan for 500,000 that is to be repaid over 15 years with level end-of-month payments calculated at an annual nominal interest rate of 10% compounded monthly.

After the 48th payment the borrower refinances the loan at a nominal interest rate of i compounded monthly. The remaining term of the loan continues for 11 years, but the new end-of-month payment is 473.98 lower than the previous payment.

Calculate i.

A 7.4% B 7.7% C 8.0% D 8.3% E 8.6%

Question 11.10

A 20-year loan for 75,000 is to be paid off with level end-of-month payments. The annual nominal interest rate on the loan is 9% compounded monthly.

Immediately after the 24th payment, the loan is refinanced at a new interest rate of 7% compounded monthly. The maturity of the loan is unchanged.

Calculate the new end-of-month payment amount.

A 559 B 588 C 617 D 646 E 675

Question 11.11

A 5-year loan of 24,000 is to be repaid with level end-of-year payments. The annual effective discount rate is 6%.

If the first four payments had been rounded up to the next multiple of 100, the final payment would be X.

Calculate X.

A 5,222 B 5,333 C 5,444 D 5,556 E 5,557

Question 11.12

A 20-year loan of 10,000 will be paid off with 20 end-of-year payments. There are two options for repaying the loan:

Level annual payments are calculated using an annual effective interest rate of 5%.

Installments of 500 at the end of each year plus the interest due on the outstanding balance. The interest is calculated using an annual effective interest rate of i.

The sum of the payments using the first option is equal to the sum of the payments using the second option.

Calculate i.

A 5.0% B 5.3% C 5.5% D 5.8% E 6.0%

Question 11.13

A 10-year loan with level end-of-year payments is taken out at an annual effective rate of interest of 6.5%. The principal portion of the 6th payment is 1,370.65.

Calculate the total amount of interest paid on the loan.

A 5,237 B 5,258 C 5,279 D 5,300 E 5,321

Question 11.14

An n-year loan is repaid with equal payments made at the end of each year.

The interest portion of the payment made at time $(n - 2)$ is equal to 0.5471 of the interest portion of the payment made at time $(n - 5)$ and is also equal to 0.2209 of the interest portion of the payment made at time 1.

Calculate n.

A 22 B 24 C 26 D 28 E 30

Question 11.15

A 30-year loan is repaid with 30 annual payments with the first payment made one year after the loan is originated.

For the first 16 years, the payments are 1,500. Thereafter, the payments decrease by 100 per year.

The principal paid with the 16th payment is equal to X.

The annual effective interest rate is i, and $v = 1/(1+i)$.

Which of the expressions below is equal to the amount of interest paid in the first year?

A $1{,}500 - X$

B Xv^{16}

C $1{,}500v^{15} - Xv^{15}$

D $15Xi$

E $1{,}500 - Xv^{15}$

Question 11.16

An n-year loan is repaid with n annual payments with the first payment made one year after the loan is originated.

For the first 20 years, the payments are 1,000. Thereafter, each payment is g% greater than its predecessor.

The principal paid with the 11th payment is equal to X.

The annual effective interest rate is i and $v = 1/(1+i)$.

Which of the expressions below is equal to the amount of interest paid in the first year?

A $1{,}000 - X$

B $Xv^{10}(1+g)$

C $1{,}000v^{10} - Xv^{10}$

D $\dfrac{10Xi}{(1+g)^{10}}$

E $1{,}000 - Xv^{10}$

Question 11.17

A 20-year loan of L is repaid with level annual payments at the end of each year.

The interest paid in the 1st year is 4,316. The principal paid in the 11th year is 4,080.

Calculate L.

A 81,000 B 81,500 C 83,000 D 83,750 E 84,000

Question 11.18

A borrower takes out a loan of 84,000 at an annual effective discount rate of 6%.

The loan is to be repaid with annual payments of 10,000.

The first payment is due four years after the loan is taken out.

Calculate the amount of the drop payment.

A 7,674 B 7,699 C 7,724 D 7,748 E 7,772

Question 11.19

On January 1, 2011 a borrower took out a 20-year mortgage loan for 400,000 at a nominal annual interest rate of 9% compounded monthly. The loan was to be repaid with level end-of-month payments with the first payment on January 31, 2011.

The borrower paid an extra 15,000 in addition to the regular monthly payment on December 31 of 2011, 2012, 2013, and 2014.

Because of the extra payments, the final payment is a drop payment. Determine the date of the drop payment.

A 1/1/25 B 3/31/25 C 6/30/25 D 9/30/25 E 12/31/25

Question 11.20

Ashley borrows 100,000 for 10 years at an annual effective interest rate of 11%.

At the end of each year, she pays the interest on the loan and also deposits a level payment into a sinking fund. The sinking fund earns an annual effective interest rate of 7%.

At the end of 10 years, the amount in the sinking fund is used to repay the loan.

Calculate the total amount of the payments made by Ashley over the 10-year period.

A 180,000 B 182,378 C 184,756 D 187,134 E 189,512

Question 11.21

Keith borrows 20,000 for 20 years at an annual effective interest rate of 5%. He can repay this loan by making level end-of-year payments of 1,604.85.

Instead, Keith uses a sinking fund to retire the loan at the end of 20 years. Until the loan is retired, he pays the interest on the loan to the lender at the end of each year and then deposits the remainder of the 1,604.85 into a sinking fund that earns an annual effective rate of 6%.

Calculate the balance remaining in the sinking fund immediately after the loan is repaid.

A 2,250	B 2,275	C 2,300	D 2,325	E 2,350

Question 11.22

A 10-year loan of 10,000 can be repaid in two ways:

The loan can be repaid using the amortization method with 10 equal annual payments computed using an annual effective interest rate of 7%.

The loan can be repaid using a sinking fund method in which the lender receives an annual effective rate of 5.5% and the sinking fund earns an annual effective interest rate of i.

Both methods require the same annual payment to be made at the end of each year for 10 years.

Calculate i.

A 3.0%	B 3.6%	C 4.3%	D 4.9%	E 5.5%

Question 11.23

A 10-year loan of 10,000 is made at an annual effective interest rate of i.

At the end of each year, the borrower pays the interest due on the loan and deposits twice the amount of the interest payment into a sinking fund that earns interest at an annual effective rate of $0.6i$. The sinking fund accumulates to the amount necessary to pay off the loan at the end of 10 years.

Calculate i.

A 4.2%	B 4.4%	C 4.6%	D 4.8%	E 5.0%

Question 11.24

A 14-year loan is to be repaid with the sinking fund method. The sinking fund earns an annual effective interest rate of 6.2%.

Immediately following the 6^{th} payment and deposit, the difference between the amount owed to the lender and the accumulated value of the sinking fund is 59,053.

Calculating the amount of each sinking fund deposit.

A 3,871	B 4,000	C 4,129	D 4,258	E 4,358

Chapter 12: Project Evaluation

12.01 Net Present Value

In the context of this chapter, a project is an investment that produces future cash flows.

Consider a project that produces positive and negative cash flows to the owner of the project. That is, the project provides cash inflows, but it also requires cash outflows.

Unless stated otherwise, a cash flow is considered positive if it is a cash inflow and negative if it is a cash outflow. This doesn't completely clear up the potential for confusion because when there are two parties to a transaction, a cash inflow to one party may be a cash outflow to the other. In this section, we are considering the perspective of the party that is investing in, or considering an investment in, a project.

The **net present value** of the project is the amount by which the present value of the cash inflows exceeds the present value of the cash outflows, where both cash flows are calculated at a required rate of return:

$$NPV_0 = PV_0(\text{Cash Inflows}) - PV_0(\text{Cash Outflows})$$

Alternatively, since the cash outflows are simply negative cash flows, we can write the net present value as:

$$NPV_0 = PV_0(\text{Cash Flows})$$

12.01

Net Present Value

The net present value of a project is the present value of a project's cash flows, discounted at the required rate of return:

$$NPV_0 = PV_0(\text{Cash Flows})$$

Example 12.01

An industrialist is considering the purchase of a new machine for \$50,000. If purchased, the machine can be used to convert \$1,000 of raw materials at the beginning of each year into widgets than can be sold for \$9,000 at the end of each year. The machine will last for 10 years, and it is worthless thereafter. The required rate of return is an annual effective rate of 8%.

Calculate the net present value of purchasing the machine and operating it for 10 years.

Solution

At the beginning of the first year, the industrialist can purchase the machine for \$50,000 and purchase raw materials for \$1,000, so the cash flow at time 0 is:

$$CF_0 = -50{,}000 - 1{,}000 = -51{,}000$$

At the end of years 1 through 9, the investor receives \$9,000 from sale of the widgets and then immediately purchases another \$1,000 of raw materials. Therefore, the net cash flow at times 1 through 9 is \$8,000:

$$CF_1 = CF_2 = \cdots = CF_9 = 9{,}000 - 1{,}000 = 8{,}000$$

At the end of 10 years, the sale of the widgets produced in the 10th year results in revenue of \$9,000. At the end of 10 years, there is no need to purchase additional raw materials, because the machine's productive lifetime is over, so the final cash flow is:

$$CF_{10} = 9{,}000$$

The net present value of the cash flows is:

$$NPV_0 = PV_0(\text{Cash Flows}) = -51{,}000 + 8{,}000a_{\overline{9}|} + 9{,}000v^{10}$$

$$= -51{,}000 + 8{,}000 \times \frac{1 - 1.08^{-9}}{0.08} + \frac{9{,}000}{1.08^{10}} =$$

$$= -51{,}000 + 8{,}000 \times 6.2469 + 4{,}168.74 = \mathbf{3{,}143.84}$$

We can also use the time value of money worksheet in the BA II Plus calculator to find the net present value:

10 [*N*] 8 [*I/Y*] 8,000 [*PMT*] [CPT] [*PV*] 1,000 [*FV*] [CPT] [PV]
+ 51,000 [=]
Result is −3,143.84. Answer is **3,143.84**.

Alternatively, we can use the cash flow worksheet in the BA II Plus calculator to find the net present value. Below, we enter the amount of each cash flow and its corresponding frequency. The initial cash flow is assumed to have a frequency of 1, so we don't enter a frequency for the initial cash flow:

[CF] CFo = −51,000 [ENTER] ↓
C01 = 8,000 [ENTER] ↓ F01 = 9 [ENTER] ↓
C02 = 9,000 [ENTER] ↓ F02 = 1
[NPV] 8 [ENTER] ↓ [CPT]
NPV = **3,143.84**

This textbook's notation for a cash flow is a little different from the notation used by the BA II Plus calculator. For example, this textbook uses CF_2 to indicate a cash flow that occurs at time 2. The BA II Plus calculator, however, uses C02 to indicate the amount of a cash flow occurring one time unit after F01 time units and continuing for F02 units of time.

12.02 Internal Rate of Return

The **internal rate of return** (IRR) is the required return that causes the net present value of a project's cash flows to be zero. The IRR is also known as the **yield** or the **dollar-weighted rate of return**.

The internal rate of return is the interest rate at which the present value of the cash inflows is equal to the present value of the cash outflows:

$$0 = PV_0(\text{Cash Inflows}) - PV_0(\text{Cash Outflows})$$

$$PV_0(\text{Cash Outflows}) = PV_0(\text{Cash Inflows})$$

Alternatively, since the cash outflows are simply negative cash flows, the IRR is the interest rate that satisfies:

$$PV_0(\text{Cash Flows}) = 0$$

12.02

Internal Rate of Return

The internal rate of return is the rate of return that results in a net present value of zero:

$$PV_0(\text{Cash Flows}) = 0$$

Example 12.02

A Susan deposits $8,000 into an investment account. After 4 years, she withdraws $5,000, and at the end of 8 years she withdraws the remaining balance of $5,414.21.

Calculate the annual effective internal rate of return.

Solution

The internal rate of return is the rate that results in the present value of the cash outflow being equal to the present value of the cash inflows:

$$8{,}000 = 5{,}000v^4 + 5{,}414.21v^8$$

$$5{,}000v^4 + 5{,}414.21v^8 - 8{,}000 = 0$$

Let's set $X = v^4$ and use the quadratic formula to find X.

$$5{,}414.21X^2 + 5{,}000X - 8{,}000 = 0$$

$$X = \frac{-5{,}000 \pm \sqrt{5{,}000^2 - 4(5{,}414.21)(-8{,}000)}}{2 \times 5{,}414.21} = 0.838561$$

We can now solve for the annual effective IRR:

$$v^4 = 0.83856$$
$$v = 0.95694$$
$$1 + i = 1.04500$$
$$i = \mathbf{4.50\%}$$

If there are more than 2 cash flows, it can be difficult to solve for the IRR exactly, but we can often use the BA II Plus to find the IRR.

Example 12.03 An industrialist is considering the purchase of a new machine for \$50,000. If purchased, the machine can be used to convert \$1,000 of raw materials at the beginning of each year into widgets than can be sold for \$9,000 at the end of each year. The machine will last for 10 years, and it is worthless thereafter.

Calculate the annual effective internal rate of return resulting from the purchase of the machine and operating it for 10 years.

Solution The net cash flows for times 0 through 10 are:

$$CF_0 = -50{,}000 - 1{,}000 = -51{,}000$$
$$CF_1 = CF_2 = \cdots = CF_9 = 9{,}000 - 1{,}000 = 8{,}000$$
$$CF_{10} = 9{,}000$$

We can use the cash flow worksheet in the BA II Plus calculator to find the IRR. Below, we enter the amount of each cash flow and its corresponding frequency. The initial cash flow is assumed to have a frequency of 1, so we don't enter a frequency for the initial cash flow:

[CF] CFo = −51,000 [ENTER] ↓

C01 = 8,000 [ENTER] ↓ F01 = 9 [ENTER] ↓

C02 = 9,000 [ENTER] ↓ F02 = 1

[IRR] [CPT]

IRR = 9.3361 Answer is **9.3361%**.

We can also find the internal rate of return of an investment fund. The initial balance is treated as a deposit, and the final balance is treated as a withdrawal (since it is available for the owner to withdraw).

In the example below, deposits to a fund are treated as negative cash flows because they require that the owner of the fund make those deposits. Likewise, withdrawals from the fund are treated as positive cash flows to the owner of the fund because the owner receives the withdrawals.

Example 12.04 An initial deposit of \$1,000 is made to an investment fund. After one year, the fund balance grows to \$1,050, and then a withdrawal of \$120 is made from the fund. At the end of two years, the fund balance is \$985.80, and then a deposit of \$10,000 is made.

At the end of 3 years, fund balance is \$13,182.96, and then a withdrawal of \$12,000 is made from the fund. At the end of four years, the fund balance is \$1,230.28.

Calculate the annual effective internal rate of return of the fund.

Solution The net cash flows for times 0 through 4 are:

$$CF_0 = -1{,}000.00$$
$$CF_1 = 120.00$$
$$CF_2 = -10{,}000.00$$
$$CF_3 = 12{,}000.00$$
$$CF_4 = 1{,}230.28$$

We can use the cash flow worksheet in the BA II Plus calculator to find the IRR. Below, we enter the amount of each cash flow and we leave its frequency set to 1. The initial cash flow is assumed to have a frequency of 1, so we don't enter a frequency for the initial cash flow:

[CF] CFo = −1,000 [ENTER] ↓
C01 = 120 [ENTER] ↓ ↓
C02 = −10,000 [ENTER] ↓ ↓
C03 = 12,000 [ENTER] ↓ ↓
C04 = 1,230.28 [ENTER]
[IRR] [CPT]
IRR = 16.4165 Answer is **16.4165%**.

In the first, second, third, and fourth years of the example above, the investment earned 5%, 6%, 20%, and 4% respectively. The amount of dollars in the fund during the third year was much higher than the amount in the fund during the other years, and that is why the IRR of 16.4165% is closer to 20% than to the other rates. Another way to say this is that the IRR is a dollar-weighted rate of return.

In the example below, the cash flows occur quarterly, so we find the quarterly effective interest rate and then convert it into the annual effective interest rate needed to answer the question.

Example 12.05 Sam has $10,000 in his bank account. At the end of 3 months, he deposits $500. At the end of 6 months, he withdraws $3,000. At the end of 9 months, he deposits $2,000. At the end of one year, the balance in the bank account is $11,000.

What is the annual effective internal rate of return earned by Sam on the bank account?

Solution The present value of the cash flows is zero:

$$-10{,}000 - \frac{500}{(1+i)^{0.25}} + \frac{3{,}000}{(1+i)^{0.5}} - \frac{2{,}000}{(1+i)^{0.75}} + \frac{11{,}000}{1+i} = 0$$

The BA II Plus can be used to find the 3-month effective interest rate:

[CF] CF0 = −10,000 [ENTER] ↓
C01 = −500 [ENTER] ↓ ↓
C02 = 3,000 [ENTER] ↓ ↓
C03 = −2,000 [ENTER] ↓ ↓
C04 = 11,000 [ENTER]
[IRR] [CPT]
IRR = 3.7722172

Converting this quarterly IRR into an annual effective IRR, we have:

$$i = 1.03772172^4 - 1 = \mathbf{0.159641}$$

When the present value of a set of cash flows is zero, the current value of the cash flows is also zero at any point in time, t:

$$PV_0(\text{Cash Flows}) = 0$$

$$(1+i)^t \times PV_0(\text{Cash Flows}) = (1+i)^t \times 0$$

$$CV_t(\text{Cash Flows}) = 0$$

Suppose that the final cash flow occurs at time n. The current value at time n can be expressed as an accumulated value:

$$CV_n(\text{Cash Flows}) = 0$$

$$AV_n(\text{Cash Flows}) = 0$$

Thus, in addition to producing a net present value of zero, the IRR also produces an accumulated value of zero.

When considering the accumulated value, it can be more intuitive to think of the deposits as positive numbers and the withdrawals as negative numbers. This approach looks at the cash flows from the perspective of the party that receives the deposits. It produces the same IRR because changing the sign on all of the terms doesn't affect the result. The example above uses the following equation when treating deposits as negative cash flows:

$$-10{,}000 - \frac{500}{(1+i)^{0.25}} + \frac{3{,}000}{(1+i)^{0.5}} - \frac{2{,}000}{(1+i)^{0.75}} + \frac{11{,}000}{1+i} = 0$$

Multiplying this equation by -1 does not change the IRR:

$$10{,}000 + \frac{500}{(1+i)^{0.25}} - \frac{3{,}000}{(1+i)^{0.5}} + \frac{2{,}000}{(1+i)^{0.75}} - \frac{11{,}000}{1+i} = 0$$

When finding the IRR of an investment fund, it's easy to become confused about which sign a cash flow should have. Keep in mind:

1. *The initial balance is treated as a deposit, and the final balance is treated as a withdrawal.*
2. *All deposits have the same sign. All withdrawals have the same sign, which is the opposite of the sign of the deposits.*

Accumulating both sides for 1 year and moving the ending balance to the right side of the equation results in:

$$10{,}000(1+i) + 500(1+i)^{0.75} - 3{,}000(1+i)^{0.5} + 2{,}000(1+i)^{0.25} = 11{,}000$$

In the form above, we see that the \$10,000 initial balance and the deposits of \$500 and \$2,000 are available to earn interest. The \$3,000 withdrawal is a reduction in the funds available to earn interest.

When the time interval over which the cash flows occur is fairly short, we can obtain an approximation to i by assuming that the interest rate is a simple interest rate:

$$10{,}000(1+i) + 500(1+0.75i) - 3{,}000(1+0.5i) + 2{,}000(1+0.25i) = 11{,}000$$

$$(10{,}000 \times 1 + 500 \times 0.75 - 3{,}000 \times 0.5 + 2{,}000 \times 0.25)i = 1{,}500$$

The value in the parentheses on the left side of the equation is the exposure of the fund to interest. \$10,000 is on deposit at the beginning of the year, so it is available to earn interest for a full year. \$500 is deposited after 3 months, and so it is available to earn interest for the remaining 0.75 years. \$3,000 is withdrawn after 6 months, reducing the balance available to earn interest for the following 0.5 years. Finally, \$2,000 is deposited after 9 months, and is therefore available to earn interest for the remaining 0.25 years. The exposure of the fund is the weighted average amount of the net deposits, weighted by the amount of time that the net deposits are in the fund.

The 1,500 on the right side of the equation above is the income produced by the fund because it is the withdrawals minus the deposits.

The withdrawals exceed the deposits by \$1,500, and the difference is income:

$$\text{Withdrawals} - \text{Deposits} = \text{Income}$$

$$11{,}000 + 3{,}000 - 10{,}000 - 500 - 2{,}000 = 1{,}500$$

We continue to treat the initial balance as a deposit and the final balance as a withdrawal. Alternatively, we could use the following formula:

$$\text{Final Balance} + \text{Withdrawals} - \text{Initial Balance} - \text{Deposits} = \text{Income}$$

When you get more out of an investment than you put into it, the difference is income!

Let's finish solving for i:

$$(10{,}000 \times 1 + 500 \times 0.75 - 3{,}000 \times 0.5 + 2{,}000 \times 0.25)i = 1{,}500$$

$$i = \frac{1{,}500}{9{,}375}$$

$$i = 0.1600$$

The actual value of i is 15.9641%, so the value of 16.0000% is fairly close to the actual value.

The simple interest approximation is useful when the cash flows are not evenly spaced because unevenly spaced cash flows are inconvenient to input into the cash flow worksheet of the BA II Plus calculator.

Some texts refer to the simple interest approximation as the dollar-weighted rate of return. More commonly, however, the dollar-weighted rate of return is defined to be equal to the IRR itself, not the approximation.

Simple Interest Approximation for the Dollar-Weighted Return

12.03 The simple interest approximation for the dollar-weighted rate of return replaces $(1+i)^k$ with $(1+ki)$ when the cash flows occur over a relatively short interval of time. The resulting value of i can be found as:

$$i = \frac{\text{Income}}{\text{Fund exposure}}$$

where:

$$\text{Income} = \text{Withdrawals} - \text{Deposits}$$

$$\text{Fund exposure} = \sum(\text{Net deposit})(\text{Time deposit is in the fund})$$

The simple interest approximation is exact when:

- There is only one deposit, which is made at time 0, and
- There is only one withdrawal, which is made at time 1.

Even when these two conditions are not met, the simple interest approximation can be quite close the actual IRR. The smaller the net deposits between time 0 and time 1 (relative to the initial deposit), the more exact the estimate becomes.

Example 12.06 The initial balance in a fund is \$5,000. Deposits of \$300 and \$1,200 are made to the fund after 1 month and 9 months, respectively. A withdrawal of \$2,000 is made after 2.5 months. At the end of 10 months, the balance in the fund is \$4,800.

Estimate the annual effective dollar-weighted rate of return earned by the fund.

Solution The income is:

$$\text{Income} = \text{Withdrawals} - \text{Deposits} = (2{,}000 + 4{,}800) - (5{,}000 + 300 + 1{,}200)$$

$$= 300$$

The exposure is:

$$\text{Fund exposure} = \sum(\text{Net deposit})(\text{Time deposit is in the fund})$$

$$= 5{,}000 \times \frac{10}{12} + 300 \times \frac{9}{12} + 1{,}200 \times \frac{1}{12} - 2{,}000 \times \frac{7.5}{12} = 3{,}241.67$$

The estimate for the IRR is:

$$i = \frac{\text{Income}}{\text{Fund exposure}} = \frac{300}{3{,}241.67} = \mathbf{0.09254}$$

The actual IRR of the fund in the example above is 0.09303.

When the cash flows of a project are almost evenly spaced, it is sometimes possible to avoid using the simple interest approximation by including a few cash flows of zero in the BA II Plus cash flow worksheet.

Example 12.07 Sarah has $10,000 in her bank account. At the end of 9 months, she deposits $500. At the end of one year, the balance in the bank account is $11,000.

What is the annual effective internal rate of return?

Solution The present value of the cash flows is zero. The cash flows are equally spaced if we include a cash flow of 0 at the end of 3 months and another cash flow of 0 at the end of 6 months:

$$-10{,}000 - \frac{0}{(1+i)^{0.25}} + \frac{0}{(1+i)^{0.5}} - \frac{500}{(1+i)^{0.75}} + \frac{11{,}000}{1+i} = 0$$

The BA II Plus can be used to find the 3-month effective interest rate:

[CF] CF0 = –10,000 [ENTER] ↓

C01 = 0 [ENTER] ↓ ↓

C02 = 0 [ENTER] ↓ ↓

C03 = –500 [ENTER] ↓ ↓

C04 = 11,000 [ENTER]

[IRR] [CPT]

IRR = 1.212607

Converting this quarterly IRR into an annual effective IRR, we have:

$$i = 1.01212607^4 - 1 = \mathbf{0.049394}$$

12.03 Time-Weighted Rate of Return

The internal rate of return is also known as the dollar-weighted rate of return because the IRR depends on the quantity of dollars deposited and withdrawn from the fund. During an interval when the fund is relatively large, there is more weight placed on that interval's rate of return, and during an interval when the fund is relatively small, there is less weight placed on the interval's rate of return.

For some investments, such as mutual funds, the fund manager does not have control over deposits into and withdrawals from the fund. Therefore, a more useful measure of the fund manager's performance is the return that would be earned if funds were deposited at the beginning of the measurement period and withdrawn only at the end of the measurement period. To determine this return, the distortion caused by intervening cash flows must be removed.

This is accomplished by measuring the accumulation factor over each interval between deposits and withdrawals. These accumulation factors are then multiplied together to produce an accumulation factor applicable to the length of the project. This accumulation factor is then converted into an interest rate, usually an annual effective interest rate. This interest rate is called the time-weighted rate of return.

The time-weighted rate of return is not affected by cash inflows and outflows. Instead it is an average (albeit, not a linear average) of the rates of return earned over the various time intervals. This average is based on the amount of time each rate is earned, which is why it is called the time-weighted rate of return.

Time-Weighted Rate of Return

12.04

Consider a fund that has net deposits of $ND_1, ND_2, \cdots, ND_n$ at times $t_1, t_2, \cdots, t_n$. The amount in the fund just before each net deposit is $F_1, F_2, \cdots, F_n$. The initial amount in the fund at time 0 is F_0. Withdrawals are recorded as negative net deposits. The effective time-weighted rate of return per unit of time is the value of i that satisfies:

$$(1+i)^{t_n} = \left(\frac{F_1}{F_0}\right)\left(\frac{F_2}{F_1 + ND_1}\right)\cdots\left(\frac{F_n}{F_{n-1} + ND_{n-1}}\right)$$

Positive values of ND indicate deposits, and negative values indicate withdrawals. The final net deposit, ND_n, is not used in the calculation of the rate of return, because it occurs at the very end of the time interval under consideration, and therefore, it has no opportunity to earn any return. In practice, the value of ND_n is usually zero.

Example 12.08

An initial deposit of $1,000 is made to an investment fund. After one year, the fund balance grows to $1,050, and then a withdrawal of $120 is made from the fund. At the end of two years, the fund balance is $985.80, and then a deposit of $10,000 is made. At the end of 3 years, the fund balance is $13,182.96, and then a withdrawal of $12,000 is made from the fund. At the end of four years, the fund balance is $1,230.28.

Calculate the annual effective time-weighted rate of return of the fund.

Solution

The time-weighted rate of return is found below:

$$(1+i)^{t_n} = \left(\frac{F_1}{F_0}\right)\left(\frac{F_2}{F_1 + ND_1}\right)\cdots\left(\frac{F_n}{F_{n-1} + ND_{n-1}}\right)$$

$$(1+i)^4 = \frac{1{,}050}{1{,}000} \times \frac{985.80}{1{,}050 - 120} \times \frac{13{,}182.96}{985.80 + 10{,}000} \times \frac{1{,}230.28}{13{,}182.96 - 12{,}000}$$

$$(1+i)^4 = 1.05 \times 1.06 \times 1.20 \times 1.04$$

$$(1+i)^4 = 1.38902$$

$$i = \mathbf{0.08562}$$

In Section 12.02, we found the IRR of the example above to be 16.4165%, which is relatively close to the 20% earned in the third year. We now find that the time-weighted rate of return is 8.562%, which is relatively close to the average of the rates of return earned in each year:

$$\frac{0.05 + 0.06 + 0.20 + 0.04}{4} = 0.0875$$

This illustrates that the dollar-weighted rate of return (or IRR) is influenced by the amount of dollars in the fund when a return is earned, while the time-weighted rate of return is influenced by how long a rate of return is earned. In the example above, the returns of 5%, 6%, 20%, and 4% were all earned for 1 year, so the time-weighted rate of return is close their weighted average, with each weight being equal.

Rather than use the formula in the Key Concept above, it is often easier to think about the rate of return earned over each interval.

If i_k is the rate of return earned over the k^{th} interval, then each of the ratios in the parentheses is an accumulation factor:

$$\begin{aligned}(1+i)^{t_n} &= \left(\frac{F_1}{F_0}\right)\left(\frac{F_2}{F_1+ND_1}\right)\cdots\left(\frac{F_n}{F_{n-1}+ND_{n-1}}\right)\\ &= \left(1+i_1\right)^{t_1}\left(1+i_2\right)^{t_2-t_1}\cdots\left(1+i_n\right)^{t_n-t_{n-1}}\end{aligned}$$

Although it is not necessary to calculate $i_1, i_2, \cdots, i_n$ to solve for i, the notation above shows that each accumulation factor depends on the rate of return and the length of time that the rate of return is earned.

Example 12.09

On January 1, a fund is worth 100,000. On March 1, the value has fallen to 90,000 and then 40,000 is deposited. On November 1, the value has increased to 180,000 and then 30,000 is withdrawn. On January 1 of the following year, the fund is worth 140,000.

Calculate the annual effective time-weighted rate of return of the fund.

Solution

The time-weighted rate of return is found below:

$$\begin{aligned}(1+i)^{t_n} &= \left(\frac{F_1}{F_0}\right)\left(\frac{F_2}{F_1+ND_1}\right)\cdots\left(\frac{F_n}{F_{n-1}+ND_{n-1}}\right)\\ (1+i)^{t_n} &= \left(1+i_1\right)^{t_1}\left(1+i_2\right)^{t_2-t_1}\cdots\left(1+i_n\right)^{t_n-t_{n-1}}\\ (1+i)^1 &= \frac{90{,}000}{100{,}000}\times\frac{180{,}000}{90{,}000+40{,}000}\times\frac{140{,}000}{180{,}000-30{,}000}\\ 1+i &= 1.16308\\ i &= \mathbf{0.16308}\end{aligned}$$

12.04 Defaults

Loans are not always repaid in full. When a borrower fails to repay a loan in full, the borrower is said to have defaulted on the loan. The default rate is the percentage of the loaned funds that are not repaid in full. The recovery rate is the percentage of the amount due that is recovered on a loan that has defaulted:

Def = Percentage of loaned funds not repaid in full

Rec = Recovery rate for loans that default

Lenders prefer low default rates and high recovery rates. Consider a group of loans, where each loan is to be repaid with a single payment of principal and interest upon maturity. Let R_{nom} be the nominal rate of the loans. The nominal rate is also known as the stated or contractual rate. Assuming that the interest rates are expressed as continuously compounded rates, the net interest rate realized by the investor, R_{net}, for a t–year loan is described by the following relationship:

$$e^{R_{net}\times t} = \left(1-Def\right)e^{R_{nom}\times t} + Def\left(Rec\right)e^{R_{nom}\times t}$$

In the expression above, (1 – *Def*) is the percentage that pays in full and *Def* is the percentage that pays only a portion of the amount due. The defaulted loans pay out the recovery rate times the amount due.

If the interest rates are expressed as annual effective rates, then:

$$\left(1+R_{net}\right)^t = \left(1-Def\right)\left(1+R_{nom}\right)^t + Def\left(Rec\right)\left(1+R_{nom}\right)^t$$

We use the word "nominal" in two different ways in this text:

1. *Nominal refers to an interest rate that is compounded some number of times per unit of time.*

2. *In this section and the next section, nominal is a descriptor that is used to describe a fixed interest rate on a loan that does not provide inflation protection for the lender.*

12.05

Net Realized Interest Rate

The net realized interest rate depends on the stated contractual interest rate, the default rate, and the recovery rate. We assume that each loan is to be repaid with a single payment of principal and interest upon maturity.

If the interest rates are continuously compounded rates, then:

$$e^{R_{net} \times t} = (1 - Def)e^{R_{nom} \times t} + Def(Rec)e^{R_{nom} \times t}$$

If the interest rates are annual effective rates, then:

$$(1 + R_{net})^t = (1 - Def)(1 + R_{nom})^t + Def(Rec)(1 + R_{nom})^t$$

Once defaults are taken into account, the lender's actual compensation for deferring consumption and making the loan is the net interest rate, not the nominal interest rate.

Example 12.10 A bank offers a 4-year loan that is to be repaid with a single payment of principal and interest at time 4. The bank seeks to earn a net interest rate of 6% compounded continuously. The percentage of borrowers that will default is 2%. For each loan that defaults, the bank will recover 65% of the amount owed in 4 years.

Calculate the continuously compounded contractual annual interest rate that the bank charges on the loans.

Solution The bank intends to earn a net interest rate of 6%. The contractual interest rate is R_{nom}:

$$e^{R_{net} \times t} = (1 - Def)e^{R_{nom} \times t} + Def(Rec)e^{R_{nom} \times t}$$

$$e^{0.06 \times 4} = (1 - 0.02)e^{R_{nom} \times 4} + 0.02(0.65)e^{R_{nom} \times 4}$$

$$e^{4R_{nom}} = 1.27125$$

$$R_{nom} = \mathbf{0.06176}$$

To compensate for the risk of default losses, lenders charge higher nominal interest rates for riskier loans.

Example 12.11 A bank offers a 4-year loan that is to be repaid with a single payment of principal and interest at time 4. The bank lends 100,000 each to Bruce and Robin.

The probability that Bruce will default on the loan is X, and the probability that Robin will default is Y. In the event of default, the recovery rate for both loans is 40%.

The bank observes that $X < Y$. Therefore, the bank charges Bruce a lower stated interest rate than the rate it charges Robin. The stated interest rate on Bruce's loan is an annual effective interest rate of 8%. The stated interest rate on Robin's loan is 10%.

The bank expects to earn a net annual effective interest rate of 7% on both loans.

Calculate the amount by which Y exceeds X.

Solution The information about the loan to Bruce can be used to find the probability that Bruce will default, X:

$$(1 + R_{net})^t = (1 - Def)(1 + R_{nom})^t + Def(Rec)(1 + R_{nom})^t$$

$$(1.07)^4 = (1 - X)(1.08)^4 + X(0.40)(1.08)^4$$

$$\left(\frac{1.07}{1.08}\right)^4 = 1 - X + 0.40X$$

$$X = 0.06088$$

The information about the loan to Robin can then be used to find the probability that Robin will default, Y:

$$(1+R_{net})^t = (1-Def)(1+R_{nom})^t + Def\,(Rec)(1+R_{nom})^t$$

$$(1.07)^4 = (1-Y)(1.10)^4 + Y(0.40)(1.10)^4$$

$$\left(\frac{1.07}{1.10}\right)^4 = 1-Y+0.40Y$$

$$Y = 0.17451$$

The difference is:

$$Y - X = 0.17451 - 0.06088 = \mathbf{0.11364}$$

The portion of the nominal interest rate that is attributable to the risk of default is called the compensation for default risk or the **credit spread**, and it is denoted by s:

$$s = \text{Compensation for default risk} = \text{Credit spread}$$

In the bond market, the credit spread is the difference between the nominal interest rate of the bond and the nominal interest rate of an otherwise equivalent risk-free bond. When the defaults are known with certainty, R_{net} is a risk-free rate, so the credit spread is the difference between the nominal interest rate and the net interest rate:

$$\text{Defaults known with certainty} \quad \Rightarrow \quad s = R_{nom} - R_{net}$$

U.S. Treasury bonds, which are considered to be free of default risk, have a credit spread of zero.

Although bonds are referenced in this chapter, they are explained in more detail in the next chapter.

Taking into account the fact that defaults are not usually known with certainty, the credit spread consists primarily of three components:

1. Expected value of default losses, based on
    - the expected default rate, and
    - the expected recovery rate
2. Compensation for accepting the risk that default losses could be greater than expected
3. Cost of evaluating the creditworthiness of the borrower

The third component refers to the cost in time and resources to evaluate a borrower's likelihood to repay a loan. Large investment companies commonly maintain credit research departments that analyze the credit worthiness of potential borrowers.

Credit spreads vary across maturities, and they usually increase with the time until maturity because the default risk increases as the time until repayment increases. A graph of credit spreads over the time remaining until maturity is known as a **spread curve**.

In the next section, we see how the credit spread is incorporated into the nominal interest rate.

12.05 Inflation

When making a loan, there is both a drawback and a motivation for the lender. The drawback is that the lender is unable to use the loaned funds to consume goods and services now. The motivation is that the lender is able to charge interest and therefore have more funds available in the future. That is, the lender receives compensation for deferring consumption. This compensation, R, is the rate of return that compensates the lender for giving up the opportunity to purchases goods and services now:

$$R = \text{Compensation for deferred consumption}$$

In the simple case of the previous section, where defaults are known with certainty and there is no inflation, the compensation for deferred consumption is equal to R_{net}.

In this section, we usually assume that R is a continuously compounded rate of return, but we may also encounter a rate of compensation for deferred consumption that is expressed as an annual effective rate.

It might seem that the interest rate for a loan would be R, but the lender must also consider the possibilities of inflation and default. Inflation is the average rate of price increase for goods and services. Not all items increase in price at the same rate, but the inflation rate remains a useful indicator of the purchasing power of a dollar. The most common way to measure inflation is to use an index to monitor the prices of a set of items. The U.S. Bureau of Labor Statistics reports the inflation rate for the Consumer Price Index (CPI) and the Producer Price Index (PPI). The CPI measures the rate at which the prices of consumer goods and services are increasing, while the PPI measures the rate at which the prices received by U.S. producers are increasing.

Nominal Return Loans and Bonds

Consider a lender that loans money at a fixed nominal interest rate. Inflation is a concern for the lender because inflation reduces the purchasing power of the future payments made to the lender. Therefore, the lender requires compensation for the expected rate of inflation:

$$I_e = \text{Compensation for expected inflation} = E\left[I_a\right]$$

where:

$$I_a = \text{Actual inflation rate (which is unknown at the outset)}$$

When determining the interest rate to charge for a loan, the lender adds the expected rate of inflation to the rate of compensation for deferred consumption. Even after adding I_e, however, the lender remains exposed to risk because the future rate of inflation is not known with certainty. Therefore, the lender also adds a spread to compensate for the possibility of unexpected inflation, which we denote by I_u.

Lenders also charge a credit spread, s, to compensate for default risk, which is the risk that the loan will not be repaid in full. Credit spreads are negatively correlated with inflation because higher inflation makes it easier for borrowers to repay fixed-rate loans, thereby reducing the likelihood of default.

A loan or bond with fixed payments is known as a nominal return loan or **nominal return bond**, and the fixed rate charged by the lender is known as the **nominal interest rate**. That is, the nominal interest rate is the interest rate that is charged when the payments are fixed at the outset. The nominal interest rate sometimes is called the stated or contractual interest rate:

$$R_{nom} = R + I_e + I_u + s$$

where:

$R_{nom} = $ Nominal interest rate

$R = $ Compensation for deferred consumption

$I_e = $ Compensation for expected inflation

$I_u = $ Compensation for unexpected inflation

$s = $ Compensation for default risk

The compensation for default risk, s, could be broken into its component pieces (as is done with inflation), but in this simplified analysis, we use s to represent all of the components of the credit spread.

The formula above for the nominal interest rate might seem to suggest that if the expected inflation rate, I_e, is significantly negative, then the nominal interest rate could also be significantly negative. It seems likely, however, that if I_e were to be significantly negative, then investors would not lend at all. Instead they would choose to earn zero percent by leaving their funds in cash. Realistically, there are storage and security costs to holding large sums of cash, so an investor might be willing to accept a slightly negative interest rate to compensate for the cost of safekeeping a stockpile of cash. However, we would not expect the nominal interest rate to be significantly negative.

12.06

Nominal Interest Rate

The nominal interest rate is the fixed interest rate on a loan or bond that does not have inflation protection. When the rates are continuously compounded, the nominal interest rate is:

$$R_{nom} = R + I_e + I_u + s$$

If the rates are expressed as annual effective rates, then the nominal interest rate is:

$$R_{nom} = (1+R)(1+I_e)(1+I_u)(1+s) - 1$$

When the inflation rate is volatile and difficult to predict, the compensation for unexpected inflation is high. At the other extreme, if the inflation rate is known with certainty, then I_u is zero.

Example 12.12 The inflation rate is known with certainty to be 2.5%. The compensation for deferred consumption is 3.5%. Consider a loan for which the compensation for default risk is 1.5%. All of the rates are expressed as continuously compounded rates.

Calculate the continuously compounded nominal interest rate on the loan.

Solution Since the inflation rate is known with certainty, the expected inflation rate is the actual inflation rate and the compensation for unexpected inflation is zero:

$$I_e = 2.5\%$$
$$I_u = 0.0\%$$

The nominal interest rate on the loan is:

$$R_{nom} = R + I_e + I_u + s = 3.5\% + 2.5\% + 0.0\% + 1.5\% = \mathbf{7.5\%}$$

Example 12.13 The inflation rate is expected to be 2.5%. To compensate for the uncertainty regarding the inflation rate, a lender charges 0.3%. The compensation for deferred consumption is 3.5%. The lender requires 1.5% as the compensation for default risk for a loan. All of the rates are expressed as continuously compounded rates.

Calculate the continuously compounded nominal interest rate on the loan.

Solution The nominal interest rate on the loan is:

$$R_{nom} = R + I_e + I_u + s = 3.5\% + 2.5\% + 0.3\% + 1.5\% = \mathbf{7.8\%}$$

Real Return Loans and Bonds

A loan or a bond can be adjusted to protect the lender from inflation. This is accomplished by increasing the principal with the rate of inflation. A **real return bond**, also known as an **inflation-indexed bond**, is a bond with such a principal adjustment.

The quoted rate, of an inflation-protected loan or bond is the **real interest rate**, and it is the rate at which the purchasing power increases if there are no defaults. The real interest rate is multiplied by the principal to produce the interest payments, and principal grows with the inflation rate, as measured by a price index. The real interest rate, R_r, can be decomposed into the compensation for deferred consumption minus the cost of inflation protection plus the credit spread:

$$R_r = R - C + s$$

where:

$$C = \text{Cost of inflation protection}$$

Inflation-protected loans and bonds pay the real interest rate, and the principal increases at the rate of inflation. Therefore, if no defaults occur, then the actual, realized rate of return, R_a, is the real interest rate plus the actual inflation rate:

$$R_a = \begin{array}{c}\text{Actual rate of return on an inflation-}\\\text{protected loan that does not default}\end{array} = R_r + I_a$$

where:

$$I_a = \text{Actual rate of inflation}$$

If the rates are expressed as annual effective rates, then the actual rate of return is:

$$R_a = (1 + R_r)(1 + I_a) - 1$$

The United State Treasury issues real return bonds called **Treasury Inflation-Protected Securities** (**TIPS**) that are free of default risk.

12.07

Real Interest Rate

The real interest rate, R_r, is the increase in purchasing power resulting from an inflation-protected investment that makes its contractual payments:

$$R_r = R - C + s$$

The actual interest rate earned on an inflation-protected investment is equal to the real interest rate plus the actual inflation rate, assuming that the loan does not default:

$$R_a = R_r + I_a$$

If the inflation rate is volatile and is difficult to predict, then the cost of inflation protection is high. At the other extreme, if the inflation rate is known with certainty, then C is zero.

Is $C = I_u$? Not necessarily. Although both are spreads that compensate for inflation uncertainty, they can be different. The participants in the markets for real return bonds and nominal return bonds are not necessarily the same, so the forces of supply and demand do not necessarily force the two values to be equal. That is, the yield forgone by the buyer of a real return bond to obtain inflation protection is not necessarily equal to the yield received by the buyer of a nominal return bond for accepting inflation risk.

Example 12.14

The cost of inflation protection for a real return loan is 0.5%, the compensation for deferred consumption is 3.5%, and the compensation for default risk is 1.5%. The rates are continuously compounded rates.

Calculate the quoted rate on the loan.

Solution

The quoted rate of an inflation-protected loan is equal to the real interest rate:

$$R_r = R - C + s = 3.5\% - 0.5\% + 1.5\% = \mathbf{4.5\%}$$

The real interest rate is added to the inflation rate to obtain the actual interest rate paid on the loan.

Example 12.15

You are given the following continuously compounded rates regarding a 3-year inflation-protected loan of 100,000 that is to be repaid with a single payment in 3 years:

- The cost of inflation protection is 0.5%.
- The compensation for deferred consumption is 3.5%
- The compensation for default risk is 1.5%.

The actual inflation rates, expressed as annual continuously compounded rates, in the first, second, and third years are 2.1%, 0.9%, and 1.8%, respectively. The loan does not default.

Calculate the payment made at the end of 3 years.

Solution

The quoted rate of the inflation-protected loan is equal to the real interest rate:

$$R_r = R - C + s = 3.5\% - 0.5\% + 1.5\% = 4.5\%$$

The actual rate in each year is the real interest rate plus the inflation rate:

$$R_a = R_r + I_a$$

First year: $4.5\% + 2.1\% = 6.6\%$

Second year: $4.5\% + 0.9\% = 5.4\%$

Third year: $4.5\% + 1.8\% = 6.3\%$

The payment made at the end of 3 years is:

$$100{,}000e^{0.066}e^{0.054}e^{0.063} = \mathbf{120{,}081.44}$$

The inflation rate can be calculated from the values of a price index. The continuously compounded annual inflation rate is the natural log of the ratio of the price index value divided by its previous value:

$$e^{I_a} = \frac{\text{PriceIndex}_t}{\text{PriceIndex}_{t-1}} \quad \Rightarrow \quad I_a = \ln\left(\frac{\text{PriceIndex}_t}{\text{PriceIndex}_{t-1}}\right)$$

Example 12.16

You are given the following continuously compounded rates regarding a 3-year inflation-protected loan of 100,000 that is to be repaid with a single payment in 3 years:

- The cost of inflation protection is 0.5%.
- The compensation for deferred consumption is 3.5%
- The compensation for default risk is 1.5%.

The observed values of the price index that is used to calculate the inflation rate are:

Time	Index
0	2,344
1	2,422
2	2,555
3	2,754

The loan does not default.

Calculate the payment made at the end of 3 years.

Solution

The quoted rate of the inflation-protected loan is equal to the real interest rate:

$$R_r = R - C + s = 3.5\% - 0.5\% + 1.5\% = 4.5\%$$

The continuously compounded inflation rates in each year are:

First year: $I_a = \ln\left(\dfrac{2,422}{2,344}\right) = 0.03273$

Second year: $I_a = \ln\left(\dfrac{2,555}{2,422}\right) = 0.05346$

Third year: $I_a = \ln\left(\dfrac{2,754}{2,555}\right) = 0.07500$

The actual rate in each year is the real interest rate plus the inflation rate:

$$R_a = R_r + I_a$$

First year: $4.5\% + 3.273\% = 7.773\%$

Second year: $4.5\% + 5.346\% = 9.846\%$

Third year: $4.5\% + 7.500\% = 12.000\%$

The payment made at the end of 3 years is:

$$100,000e^{0.0773}e^{0.09846}e^{0.12000} = \mathbf{134,473.31}$$

Alternatively, we can use the real rate of return for 3 years and the ratio of the final value to the first value of the price index to find the value of the payment:

$$100,000e^{0.045\times 3} \times \frac{2,754}{2,344} = \mathbf{134,473.31}$$

The nominal interest rate, the real interest rate, and the credit spread can be observed in the marketplace, but many of the components of interest rates cannot be observed:

Observable: R_{nom}, R_r, and s

Not observable: R, I_e, I_u, and C

We cannot say for sure what the values of the unobservable components are, but it is helpful to understand how they affect the nominal and real interest rates. The actual rate of inflation, I_a, is not observable in advance, but it is observable in hindsight.

Although the expected rate of inflation, I_e, cannot be directly observed, some pundits refer to the difference between the stated yields of nominal return Treasury bonds and the quoted yields of TIPS as the expected rate of inflation. Since the credit spread of all Treasury bonds is zero, it is suggested that expected inflation is the only difference between the yields. When we compare the yields, however, we see that the spread also includes the compensation for unexpected inflation and the cost of inflation protection:

For Treasury securities: $R_{nom} - R_r = R + I_e + I_u - [R - C] = I_e + I_u + C$

Therefore, we conclude that the difference between the yields of nominal return Treasury bonds and TIPS overestimates the expected rate of inflation:

$$R_{nom} - R_r \geq I_e$$

Example 12.17

You are given the following continuously compounded rates:

- The inflation rate is expected to be 2.5%.
- The compensation for unexpected inflation is 0.3%
- The compensation for deferred consumption is 3.5%.
- The compensation for default risk is 0.0%.
- The cost of inflation protection for real return bonds is 0.5%.

Calculate the amount by which the nominal interest rate exceeds the real interest rate.

Solution The nominal interest rate is:

$$R_{nom} = R + I_e + I_u + s = 3.5\% + 2.5\% + 0.3\% + 0.0\% = 6.3\%$$

The real interest rate is:

$$R_r = R - C + s = 3.5\% - 0.5\% + 0.0\% = 3.0\%$$

The difference is:

$$R_{nom} - R_r = 6.3\% - 3.0\% = \mathbf{3.3\%}$$

Alternatively, we can calculate the difference as the sum of the expected inflation rate, the compensation for unexpected inflation, and the cost of inflation protection:

$$R_{nom} - R_r = I_e + I_u + C = 2.5\% + 0.3\% + 0.5\% = \mathbf{3.3\%}$$

Many of the topics covered in this section are covered in the Society of Actuaries' "Determinants of Interest Rates" study note by Michael A. Bean. The table below may be helpful to those that would like to cross-reference the notation in this section with the notation in that study note.

Description	**This Text**	**SOA Study Note**
Compensation for deferred consumption	R	r
Expected inflation rate	I_e	i_e
Actual inflation rate	I_a	i_a
Compensation for unexpected inflation	I_u	i_u
Nominal interest rate	R_{nom}	R^*
Compensation for default risk (credit spread)	s	s
Real interest rate*	R_r	R_1
Cost of inflation protection	C	c
Actual return of an inflation-protected investment	R_a	$R_1^{(a)}$

* The real interest rate, R_1 in the study note differs from R_r in this text in that R_1 is the real interest rate for a loan that does not have any risk of default.

12.06 Questions

Question 12.01

An investor pays 100,000 for a 5-year investment that produces cash flows of 70,000 at the end of 3 years and at the end of 4 years. The cash flows are reinvested at an annual effective rate of interest of 3%.

Using an annual effective interest rate of 7%, calculate the net present value of the 5-year investment.

A 0 B 4,355 C 4,636 D 4,917 E 5,198

Question 12.02

The net present values of the following two projects are equal at an annual effective interest rate of 8%.

Project A requires an investment of 5,000 today. The investment pays 2,000 in one year and 5,000 in two years.

Project B requires an investment of X two years from today. The investment pays 2,000 now and 5,000 in one year.

Calculate X.

A 5,000 B 5,468 C 5,936 D 6,405 E 6,873

Question 12.03

Susan receives cash flows of 100 today, 100 in one year, and 200 in two years. The present value of the cash flows is 368.15 at an annual effective interest rate of i. Calculate i.

A 5.0% B 6.0% C 7.0% D 8.0% E 9.0%

Question 12.04

At a nominal interest rate of i convertible semiannually, an investment of 1,000 immediately and 1,400 in one year accumulate to 3,000 at the end of 2 years.

Calculate i.

A 16.0% B 16.2% C 16.4% D 16.6% E 16.8%

Question 12.05

At the beginning of the year, an investment fund was established with an initial deposit of 1,000. Three months later, a deposit of 1,200 was made. Withdrawals of 300 and 600 were made at the end of 6 months and 8 months, respectively. The balance in the fund at the end of the year was 1,400.

Calculate the dollar-weighted (money-weighted) yield rate earned by the fund during the year.

A 5.7% B 5.9% C 6.1% D 6.3% E 6.5%

Question 12.06

Jill had an account with a balance of 100 on January 1 and 115 on December 31. At the end of every month during the year, she deposited 20. She made withdrawals of 15 on February 28, 10 on June 30, 120 on September 15, and 95 on October 31.

Calculate the dollar-weighted rate of return for the year.

A 10.6% B 10.8% C 11.0% D 11.2% E 11.4%

Question 12.07

On January 1, a fund is worth 80,000. On May 15, the fund is worth 100,000, and then a withdrawal of 10,000 is made. On August 23, the value has fallen to 80,000, and then 15,000 is deposited. On December 31, the fund is worth 90,000.

Calculate the time-weighted rate of return for the year.

A 5.3% B 5.5% C 5.7% D 5.9% E 6.1%

Question 12.08

On January 1, a fund was opened with an initial deposit of 10,000. On July 1, the balance was 10,800, and then an additional 4,200 was deposited.

The annual effective yield for this fund was 6.5% over the calendar year.

Calculate the time-weighted rate of return for the year.

A 7.3% B 7.5% C 7.7% D 7.9% E 8.1%

Question 12.09

You are given the following information about an investment fund:

On January 1, the value of the fund is 16.

On July 1, prior to any deposits being made, the value of the fund is 20.

On July 1, a deposit of D is made.

On December 31, the value of the fund is D.

For the year, the time-weighted return on the fund is 0%, and the dollar-weighted rate of return (using the simple interest approximation) is R.

Calculate R.

A –29% B –15% C 0% D 15% E 29%

Question 12.10

You are given the following information about an investment fund:

On January 1, the value of the fund is 100,000.

At time t, in years ($0 < t < 7/12$), the value has increased to 104,000, and 15,000 is then deposited.

On August 1, the value has increased to 124,000, and 20,000 is then withdrawn.

On December 31, the value of the fund is 110,000.

For the year, the dollar-weighted return of return (using the simple interest approximation) is equal to the time-weighted rate of return.

Calculate t.

A 0.15 B 0.18 C 0.21 D 0.24 E 0.27

Question 12.11

You are given the following information about two investment funds:

Fund A			
Date	Fund Value (before activity)	Activity	
		Deposit	Withdrawal
January 1	130		
April 1	125		$8X$
July 1	140	$2X$	
December 31	140		

Fund B			
Date	Fund Value (before activity)	Activity	
		Deposit	Withdrawal
January 1	118		
August 1	130		$5X$
December 31	128		

Over the course of the year, the dollar-weighted rate of return of Fund A is equal to i, and the time-weighted rate of return of Fund B is also equal to i.

Calculate i.

A 12.5% B 12.7% C 12.9% D 13.1% E 13.3%

Question 12.12

A bank offers a five-year loan that is to be repaid with a single payment of principal and interest in five years. The bank wants to realize an annual rate of 6% compounded continuously to compensate it for deferred consumption. The percentage of borrowers that will default on this loan is 0.9%. When the loan becomes due in five years, the bank will recover 35% of the amount owed for each loan that defaults.

The continuously compounded credit spread, expressed as an annual rate that is continuously compounded, that the bank needs to charge is s.

Calculate $10{,}000s$.

A 6 B 9 C 12 D 15 E 18

Question 12.13

A bank offers a three-year loan and a five-year loan, each of which is to be repaid upon maturity with a single payment of principal and interest. For both loans, the bank wants to realize an annual rate of 6% compounded continuously to compensate it for deferred consumption.

The percentage of borrowers that will default on the three-year loan is 0.3%, and the percentage of borrowers that will default on the five-year loan is 0.9%. When the loans become due, the bank will recover 35% of the amount owed for each loan.

The difference between the continuously compounded annual credit spreads of the two loans is X.

Calculate $10{,}000X$.

A 5 B 6 C 7 D 8 E 9

Question 12.14

A lender makes a 3-year inflation-protected loan of 200,000 to a risk-free borrower. The annual continuously compounded interest rate on the loan is 2.8% plus the rate of inflation. The loan is to be repaid with a single payment that consists of interest and principal.

The actual annual inflation rates over the subsequent three years, expressed as continuously compounded rates, are 1%, 2%, and 4%, respectively.

Calculate the amount that the borrower owes the lender at the end of three years.

A 220,593 B 230,977 C 232,790 D 233,298 E 235,643

Question 12.15

You are given the following continuously compounded rates:

- The cost of inflation protection for a real return loan is 0.6%.
- The compensation for deferred consumption is 4.0%.
- The compensation for default risk is 0.8%.
- The compensation for expected inflation is 1.5%.
- The compensation for unexpected inflation is 0.4%.

Calculate the continuously compounded stated interest rate on a nominal return bond.

A 3.2% B 4.2% C 5.6% D 6.7% E 7.3%

Question 12.16

You are given the following continuously compounded rates:

- The cost of inflation protection for a real return loan is 0.6%.
- The compensation for deferred consumption is 4.0%.
- The compensation for default risk is 0.8%.
- The compensation for expected inflation is 1.5%.
- The compensation for unexpected inflation is 0.4%.

Calculate the continuously compounded quoted interest rate on a real return bond.

A 3.2% B 4.2% C 5.6% D 6.7% E 7.3%

Question 12.17

You are given the following continuously compounded rates:

- The cost of inflation protection for a real return loan is 0.6%.
- The compensation for deferred consumption is 4.0%.
- The compensation for default risk is 0.8%.
- The compensation for expected inflation is 1.5%.
- The compensation for unexpected inflation is 0.4%.

A one-year real return loan is made, and the actual inflation rate, expressed as an annual continuously compounded rate, turns out to be 3.1%.

Calculate the actual realized rate earned by the real return loan, expressed as a continuously compounded rate.

A 3.2% B 4.2% C 5.6% D 6.7% E 7.3%

Question 12.18

You are given the following continuously compounded rates:

- The cost of inflation protection for a real return loan is 0.6%.
- The compensation for deferred consumption is 4.0%.
- The compensation for default risk is 0.8%.
- The compensation for expected inflation is 1.5%.
- The compensation for unexpected inflation is 0.4%.

A three-year, real return loan of 100,000 is made. The actual inflation rates, expressed as continuously compounded annual rates, during the subsequent three years turn out to be 1.2%, 1.9%, and 2.8%, respectively.

The loan is repaid in full at the end of three years with a single payment. Calculate the amount of the single payment.

A 110,628 B 113,428 C 117,468 D 119,602 E 120,322

Question 12.19

You are given the following continuously compounded rates:

- The cost of inflation protection for a real return loan is 0.6%.
- The compensation for deferred consumption is 4.0%.
- The compensation for default risk is 0.8%.
- The compensation for expected inflation is 1.5%.
- The compensation for unexpected inflation is 0.4%.

A three-year, nominal return loan of 100,000 is made to Anita.

A three-year, real return loan of 100,000 is made to Dave. The actual inflation rates, expressed as continuously compounded annual rates, during the subsequent three years turn out to be 1.2%, 1.9%, and 2.8%, respectively.

Each loan is repaid in full at the end of three years with a single payment. Calculate the amount by which Anita's payment exceeds Dave's payment.

A 200 B 482 C 1,005 D 1,613 E 1,941

Question 12.20

You are given the following continuously compounded rates:

- The cost of inflation protection for a real return loan is 0.6%.
- The compensation for deferred consumption is 4.0%.
- The compensation for default risk is 0.8%.
- The compensation for expected inflation is x%.
- The compensation for unexpected inflation is 0.4%.

A three-year, nominal return loan of 100,000 is made to Anita.

A three-year, real return loan of 100,000 is made to Dave. The actual inflation rates, expressed as continuously compounded annual rates, during the subsequent three years turn out to be 1.2%, 1.9%, and 2.8%, respectively.

Each loan is repaid in full at the end of three years with a single payment. The amount paid by Anita is equal to the amount paid by Dave.

Calculate x.

A 0.44 B 0.97 C 1.14 D 1.33 E 1.97

Chapter 13: Fixed Income Securities

13.01 Pricing Noncallable Bonds

In this section, we consider the price of a bond that cannot be redeemed prior to its maturity. These bonds are also known as **noncallable bonds**.

A corporation issues bonds as a way of borrowing funds. When the corporation issues a bond, it agrees to pay a **redemption value** R at the end of n units of time, and it also agrees to pay a **coupon payment** *Coup* equal to the **coupon rate** c times the **Face value** F (also known as the **par value**) at the end of each unit of time until the bond matures at time n:

$$Coup = F \times c$$

The time until the bond matures is also known as the **term** of the bond.

Unless instructed otherwise, we assume that the redemption amount is equal to the face value:

$$R = F$$

When the redemption amount is equal to the par value (or face value), the bond is sometimes referred to as a ***par value bond****. This is not to be confused with a par bond, which is a bond that is priced at par:*

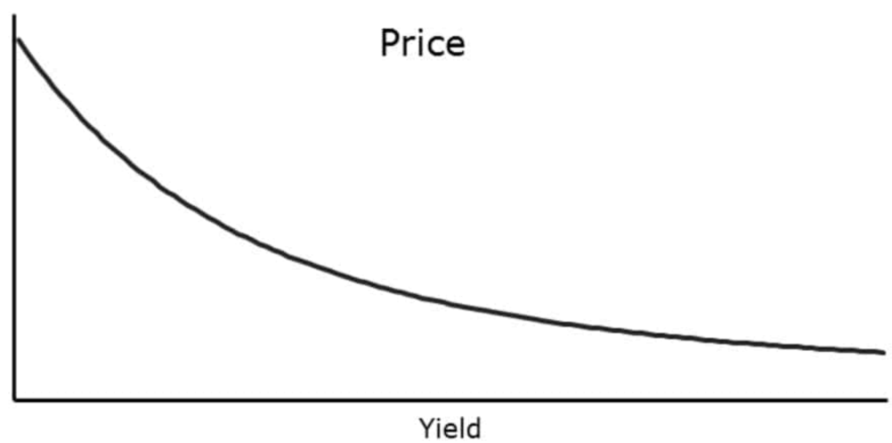
Par value bond $\Rightarrow$ *Redemption value = Par*

Par bond $\Rightarrow$ *Price = Par*

The price of the bond is the present value of the coupon payments and the redemption value, discounted at the bond's yield, which is denoted by y. Bond prices move inversely to bond yields. The higher the bond price, the lower the bond yield, and vice versa:

$$\uparrow P \Leftrightarrow \downarrow y \qquad \text{and} \qquad \downarrow P \Leftrightarrow \uparrow y$$

The graph below shows that as the yield on a bond increases, its price decreases:



13.01

Bond Price

The price of a bond that matures at the end of n units of time is:

$$P = Coup \times a_{\overline{n|}y} + Rv^n$$

where:

$P =$ Price of the bond

$R =$ Redemption value at time n

$Coup = F \times c =$ Coupon per unit of time

$y =$ Effective yield per unit of time

$$v = \frac{1}{1+y}$$

It is common for bonds to make payments semiannually.

Example 13.01 A 5-year bond makes semiannual coupon payments at an annual rate 10% per year. The coupon payments are based on a par value of $1,000, and the redemption amount is $1,100. The yield, which is compounded semiannually, is 7% per year.

What is price of the bond?

Solution Since the payments are made semiannually, the unit of time is 6 months:

$$c = \frac{0.10}{2} = 0.05$$

$$y = \frac{0.07}{2} = 0.035$$

$$n = 5 \times 2 = 10$$

The price of the bond is:

$$P = Coup \times a_{\overline{n|}y} + Rv^n = 0.05 \times 1{,}000 \times a_{\overline{10|}0.035} + \frac{1{,}100}{1.035^{10}}$$

$$= 50 \times \frac{1 - 1.035^{-10}}{0.035} + 779.81 = 50 \times 8.3166 + 779.81 = \mathbf{1{,}195.64}$$

The BA II Plus can be used to find the price:

10 [*N*]   3.5 [*I/Y*]   50 [*PMT*]   1,100 [*FV*]

[CPT] [*PV*]

Result is −1,195.64.   Answer is **1,195.64**.

We can also use the BA II Plus's bond worksheet to find the price of the bond. The calculator's worksheet always assumes that the par amount is $100, so we use the bond worksheet to find the price per $100 of par, and then we multiply by 10:

[2nd] [BOND] 12.3100 [ENTER] ↓

10 [ENTER] ↓   (enter the coupon rate)

12.3105 [ENTER] ↓

110 [ENTER] ↓

(Use [2nd] [SET] until display shows 360) ↓

(Use [2nd] [SET] until display shows 2/Y) ↓

7 [ENTER] ↓

[CPT] × 10 [=]

Result is **1,195.64.**

Although we obtained the correct price using the bond worksheet, it is often more convenient to use the TVM worksheet as shown in the example above.

When the bond yield is equal to the coupon rate and the redemption value is equal to the par value, the price of the bond is equal to the par value, and the bond is said to be priced at par:

$$P = Coup \times a_{\overline{n}|y} + Rv^n = Fya_{\overline{n}|} + Fv^n = Fy\frac{1-v^n}{y} + Fv^n$$

$$= F(1-v^n) + Fv^n = F = \text{Par Value}$$

Example 13.02 A 5-year bond makes semiannual coupon payments at an annual rate 10% per year. The par value is $1,000. The yield, which is compounded semiannually, is 10% per year.

What is price of the bond?

Solution Since the payments are made semiannually, the unit of time is 6 months:

$$c = \frac{0.10}{2} = 0.05$$

$$y = \frac{0.10}{2} = 0.05$$

$$n = 5 \times 2 = 10$$

When we are not given that a redemption value, we assume that it is equal to the par value:

$$R = 1{,}000$$

The price of the bond is:

$$P = Coup \times a_{\overline{n}|y} + Rv^n = 0.05 \times 1{,}000 \times a_{\overline{10}|0.05} + \frac{1{,}000}{1.05^{10}}$$

$$= 50 \times \frac{1-1.05^{-10}}{0.05} + 613.91 = 50 \times 7.7217 + 613.91 = \mathbf{1{,}000.00}$$

The BA II Plus can be used to find the price:

10 [*N*] 5 [*I/Y*] 50 [*PMT*] 1,000 [*FV*]

[CPT] [*PV*]

Result is −1,000.00. Answer is **1,000.00**.

We calculated the price of the bond above because it is useful to demonstrate that when the yield of a bond is equal to its coupon rate and the bond is redeemed for par, the price of the bond must be equal to the par value. Now that we know this, we could have answered the question in the example above without taking the time to do the calculations. That is, by inspection, we can see that the price of the bond must be $1,000.

If the price of bond is greater than its redemption value, then the bond is said to be priced at a **premium**. The premium is equal to the amount by which the price exceeds the redemption value:

$$P - R = \text{Premium}$$

The premium on a bond can be viewed as the present value of the extra interest provided by a bond that has a coupon that is greater than Ry. This is evident in the final expression below:

$$\text{Premium} = P - R = Coup \times a_{\overline{n}|y} + Rv^n - R = Coup \times a_{\overline{n}|y} - R(1-v^n)$$

$$= Coup \times a_{\overline{n}|y} - Rya_{\overline{n}|y} = (Coup - Ry)a_{\overline{n}|y}$$

The premium can be added to the redemption value to give us another formula for the price of a bond:

$$P = (Coup - Ry)a_{\overline{n}|y} + R$$

If the price of a bond is less than its redemption value, then the bond is said to be priced at a **discount**. The discount is equal to the amount by which the redemption value exceeds the price, and it is the opposite of the premium. The discount on a bond can be viewed as the present value of the additional interest that would be required for the bond to be priced at par. As shown in the rightmost expression below, each coupon payment would need to be increased by $Ry - Coup$ for the bond to be priced at par:

$$\text{Discount} = R - P = (Ry - Coup)a_{\overline{n|}y}$$

If the premium of a bond is negative, then the bond is priced at a discount. The discount is the opposite of the premium, and vice versa:

$$\text{Premium} = -\text{Discount}$$

13.02

Premium and Discount

The bond premium is the amount by which the price of the bond exceeds its redemption value, and it is equal to the present value of the excess of the coupon payments over Ry:

$$\text{Premium} = P - R = (Coup - Ry)a_{\overline{n|}y}$$

The bond discount is the amount by which the redemption value of the bond exceeds its price, and it is equal to the present value of the excess of Ry over the coupon payments:

$$\text{Discount} = R - P = (Ry - Coup)a_{\overline{n|}y}$$

The redemption value is often equal to the par value. When this is the case, we can write the premium of the bond as:

$$\text{Premium} = (Coup - Ry)a_{\overline{n|}y} = (Fc - Fy)\frac{1 - v^n}{y} = F\left(\frac{c - y}{y}\right)(1 - v^n)$$

Consider the three factors in the rightmost expression above. Since F and $(1 - v^n)$ are always positive, the premium is positive if $(c - y) / y$ is positive. The implication is that if $R = F$, then a coupon rate that is greater than the yield results in the bond being a premium bond, a coupon rate that is equal to the yield results in the bond having a price of par, and a coupon rate that is less than the yield results in the bond being a discount bond. This is summarized below:

When $R = F$:

$c > y \Leftrightarrow$ Premium bond

$c = y \Leftrightarrow$ Par bond

$c < y \Leftrightarrow$ Discount bond

Example 13.03 An 8-year bond makes annual coupon payments at an annual rate 10% per year. The par value is $100,000. The annual effective yield is 12% per year.

What is price of the bond?

Solution Since the payments are made annually, the unit of time is 1 year:

$c = 0.10$

$y = 0.12$

$n = 8$

When we are not given that a redemption value, we assume that it is equal to the par value:

$R = 100{,}000$

The price of the bond is:

$$P = Coup \times a_{\overline{n}|y} + Rv^n = 0.10 \times 100{,}000 \times a_{\overline{8}|0.12} + \frac{100{,}000}{1.12^8}$$

$$= 10{,}000 \times \frac{1-1.12^{-8}}{0.12} + 40{,}388.32 = 10{,}000 \times 4.9676 + 40{,}388.32$$

$$= \mathbf{90{,}064.72}$$

The BA II Plus can be used to find the price:

8 [*N*]   12 [*I/Y*]   10,000 [*PMT*]   100,000 [*FV*]

[CPT] [*PV*]

Result is −90,064.72.   Answer is **90,064.72**.

Since the yield of the bond in the example above is greater than the coupon rate, the bond is a discount bond, so it is not surprising that its price is less than its redemption value.

A portfolio of bonds consists of multiple bonds, and we can find the **portfolio yield**, which is the internal rate of return of the portfolio.

Example 13.04

Three bonds are purchased to create a bond portfolio. The time until maturity, the coupon rate, and the price of each bond are shown below:

Maturity	Coupon	Price
5	6%	$1,000.00
10	9%	$1,140.47
15	0%	$ 315.24

The 5-year and the 10-year bonds make annual payments, and all three bonds have par values of $1,000.

What is the portfolio yield of the portfolio?

Solution

The cost of the portfolio is:

$$1{,}000.00 + 1{,}140.47 + 315.24 = 2{,}455.71$$

We sum the cash flows received at the end of each year from each bond. The rightmost column below contains the sums:

Year	**5-Yr**	**10-Yr**	**15-Yr**	**Sum**
1	60.00	90.00	0.00	150.00
2	60.00	90.00	0.00	150.00
3	60.00	90.00	0.00	150.00
4	60.00	90.00	0.00	150.00
5	1,060.00	90.00	0.00	1,150.00
6		90.00	0.00	90.00
7		90.00	0.00	90.00
8		90.00	0.00	90.00
9		90.00	0.00	90.00
10		1,090.00	0.00	1,090.00
11			0.00	0.00
12			0.00	0.00
13			0.00	0.00
14			0.00	0.00
15			1,000.00	1,000.00

The BA II Plus can be used to find the internal rate of return of the portfolio. Below we enter the cost of the portfolio and the sums from the rightmost column above into the cash flow worksheet:

[CF]   CFo = −2,455.71 [ENTER] ↓

C01 = 150 [ENTER] ↓   F01 = 4 [ENTER] ↓

C02 = 1,150 [ENTER] ↓ F02 = 1 [ENTER] ↓
C03 = 90 [ENTER] ↓ F03 = 4 [ENTER] ↓
C04 = 1,090 [ENTER] ↓ F04 = 1 [ENTER] ↓
C05 = 0 [ENTER] ↓ F05 = 4 [ENTER] ↓
C06 = 1,000 [ENTER] ↓ F06 = 1 [ENTER]
[IRR] [CPT]
IRR = 7.03713 Answer is **7.03713%**.

13.02 Book Values, Book Yield, and Investment Income

Once a bond is purchased, we can use its original yield to keep track of its value. This yield is called the **book yield**, and the bond values based on it are **book values**.

The market yield of a bond can change during the life of the bond, and therefore the market value of the bond can change as well. This section deals with book values, not market values. The book values are useful for tracking the value of a bond within an accounting framework.

The initial book value is equal to the price at which the bond is purchased:

$$P = Coup \times a_{\overline{n}|y} + Rv^n$$

To find the current value, at time t, of the bond payments, we multiply both sides of the equation by $(1+y)^t$:

$$CV_t(P) = CV_t\left[Coup \times a_{\overline{n}|y} + Rv^n\right]$$

$$P(1+y)^t = (1+y)^t\left[Coup \times a_{\overline{n}|y} + Rv^n\right]$$

$$P(1+y)^t = Coup \times s_{\overline{t}|y} + Coup \times a_{\overline{n-t}|y} + Rv^{n-t}$$

Subtracting the current value of the coupons paid on or before time t from the left side we have:

$$P(1+y)^t - Coup \times s_{\overline{t}|y} = Coup \times a_{\overline{n-t}|y} + Rv^{n-t}$$

The book value at time t is equal the remaining cash flows, discounted at the book yield, y. This is the right side of the equation above, and the use of this equation to find the book value is known as the prospective method:

$$BV_t = PV_t(\text{Remaining coupon payments}) + PV_t(\text{Redemption value})$$

$$= Coup \times a_{\overline{n-t}|y} + Rv^{n-t}$$

where:

BV_t = Book value at time t

The left side of the equation provides the retrospective method for finding the book value of the bond as the accumulated price of the bond minus the accumulated value of the prior coupons:

$$BV_t = AV_t(\text{Price}) - AV_t(\text{Prior coupon payments}) = P(1+y)^t - Coup \times s_{\overline{t}|y}$$

If we know the book value at time t, then we can use a variation on the retrospective formula to find the book value at time $(t+k)$, based on the book value at time t:

$$\begin{aligned} BV_{t+k} &= P(1+y)^{t+k} - Coup \times s_{\overline{t+k}|y} \\ &= P(1+y)^t(1+y)^k - Coup\left[(1+y)^k s_{\overline{t}|y} + s_{\overline{k}|y}\right] \\ &= P(1+y)^t(1+y)^k - Coup(1+y)^k s_{\overline{t}|y} - Coup \times s_{\overline{k}|y} \\ &= \left[P(1+y)^t - Coup \times s_{\overline{t}|y}\right](1+y)^k - Coup \times s_{\overline{k}|y} \\ &= BV_t(1+y)^k - Coup \times s_{\overline{k}|y} \end{aligned}$$

The investment income, which can be reported on an income statement, is equal to the book value at the beginning of a period times the book yield:

$$InvInc_t = BV_{t-1} \times y$$

where:

$InvInc_t$ = Investment income earned in the t^{th} unit of time

Another way to calculate the investment income is to add the coupon payment to the change in the book value:

$$InvInc_t = Coup + BV_t - BV_{t-1}$$

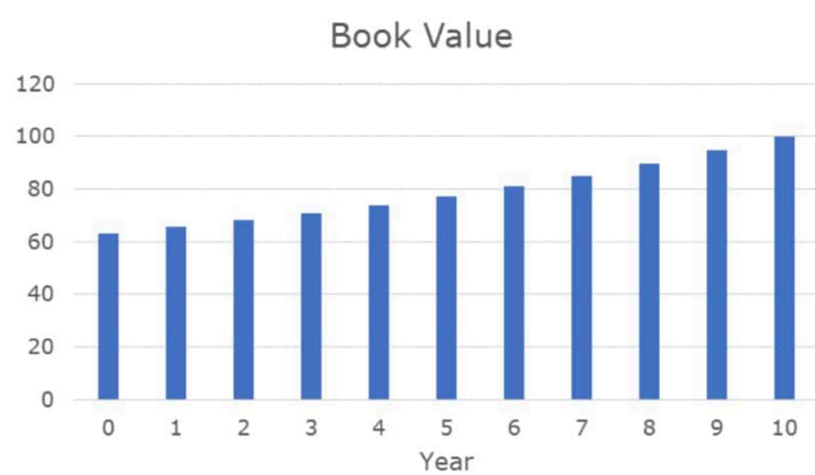
The investment income is sometimes referred to as the interest on the bond because, from the bond owner's perspective, it is equal to the interest earned, with the interest rate being equal to the book yield. From the bond issuer's perspective, however, the interest on the bond is equal to the coupon.

As the book value of a discount bond increases over time, the increase in its book value is known as the **accumulation of discount**, which is also known as the accrual of discount or the accretion of discount. Likewise, the decrease in the book value of a premium bond over time is known as the **amortization of premium**:

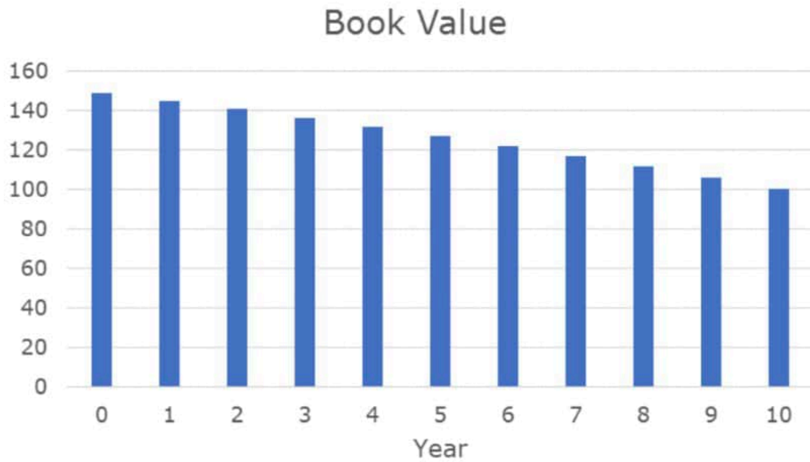
DA_t = Discount accrued in the t^{th} unit of time $= BV_t - BV_{t-1}$

PA_t = Premium amortized in the t^{th} unit of time $= BV_{t-1} - BV_t$

The graph below shows the pattern of book values of a 10-year discount bond that accrues to its redemption value of 100 at the end of 10 years.



The graph below shows the pattern of book values of a 10-year premium bond that amortizes to its redemption value of 100 at the end of 10 years.



13.03

Book Values

The book value at time t of a bond can be found as follows:

$$BV_t = Coup \times a_{\overline{n-t}|y} + Rv^{n-t} \quad \text{(Prospective Method)}$$

$$BV_t = P(1+y)^t - Coup \times s_{\overline{t}|y} \quad \text{(Retrospective Method)}$$

$$BV_t = BV_{t-1} + InvInc_t - Coup$$

$$BV_{t+k} = BV_t(1+y)^k - Coup \times s_{\overline{k}|y}$$

$$BV_t = BV_{t-1} + DA_t$$

$$BV_t = BV_{t-1} - PA_t$$

The premium amortization can be written as:

$$\begin{aligned} PA_t &= BV_{t-1} - BV_t = BV_{t-1} - (BV_{t-1} + InvInc_t - Coup) = Coup - InvInc_t \\ &= Coup - yBV_{t-1} = Coup - y\left[Coup \times a_{\overline{n-t+1}|y} + Rv^{n-t+1}\right] \\ &= Coup\left[1 - y \times a_{\overline{n-t+1}|y}\right] - Ryv^{n-t+1} = Coup\left[1 - (1 - v^{n-t+1})\right] - Ryv^{n-t+1} \\ &= Coup \times v^{n-t+1} - Ry \times v^{n-t+1} = (Coup - Ry)v^{n-t+1} \end{aligned}$$

How does the final expression above relate to the formula below that we saw in Section 11.02 regarding the principal component of a loan payment?

$$Prn_t = Pmt \times v^{n-t+1}$$

Although we don't derive the formula here, when a loan is repaid with n level payments and one final payment of R at time n, the portion of the t^{th} payment that reduces the loan amount is given by the general formula below:

$$Prn_t = \left[LevelPayment - R \times i\right]v^{n-t+1}$$

In the bond model, the book value is the amount of the loan. The level payment is Coup, the redemption value is R, and the loan interest rate is the book yield. Putting these values into the general form of the formula above gives us the portion of the book value repaid with the t^{th} level payment:

$$\textit{For bonds:} \quad PA_t = (Coup - Ry)v^{n-t+1}$$

When a loan is repaid entirely by level payments, we denote the level payments as Pmt, and there is no additional final payment so R is zero. Putting these values into the general form of the formula gives us the formula that we derived earlier in Section 11.02:

$$\textit{For loans}: \quad Prn_t = Pmt \times v^{n-t+1}$$

The ratio of the premium amortization in a coupon payment to the premium amortization in a prior coupon payment is equal to an accrual factor for the intervening length of time:

$$\frac{PA_{t+k}}{PA_t} = \frac{(Coup - Ry)v^{n-t-k+1}}{(Coup - Fy)v^{n-t+1}} = v^{-k} = (1+y)^k$$

13.04

Investment Income

The investment income earned during the t^{th} unit of time can be found as follows:

$$InvInc_t = BV_{t-1} \times y$$
$$InvInc_t = Coup + BV_t - BV_{t-1}$$
$$InvInc_t = Coup + DA_t$$

The premium amortization over the t^{th} unit of time can be described in multiple ways:

$$PA_t = BV_{t-1} - BV_t$$
$$PA_t = Coup - InvInc_t$$
$$PA_t = (Coup - Ry)v^{n-t+1}$$
$$PA_{t+k} = PA_t(1+y)^k$$
$$PA_t = -DA_t$$

An amortization schedule lists the investment income, premium amortization or discount accrual, and book value corresponding to each payment period.

Example 13.05

A 3-year bond makes semiannual coupon payments at an annual rate of 6% per year. The par value is \$100,000. The yield is 4% per year, compounded semiannually.

Prepare an amortization schedule for the bond.

Solution

Since the payments are made semiannually, the unit of time is 6 months:

$$c = \frac{0.06}{2} = 0.03 \qquad y = \frac{0.04}{2} = 0.02 \qquad n = 3 \times 2 = 6$$

The price of the bond is:

$$P = Coup \times a_{\overline{n}|y} + Rv^n = 0.03 \times 100{,}000 \times a_{\overline{6}|0.02} + \frac{100{,}000}{1.02^6}$$
$$= 3{,}000 \times \frac{1 - 1.02^{-6}}{0.02} + 88{,}797.14 = 3{,}000 \times 5.6014 + 88{,}797.14$$
$$= 105{,}601.43$$

Each row of the amortization table can be found with the following formulas:

$$InvInc_t = BV_{t-1} \times y$$
$$PA_t = Coup - InvInc_t$$
$$BV_t = BV_{t-1} - PA_t$$

The first full row of the amortization table is:

$$InvInc_1 = BV_0 \times y = 105{,}601.43 \times 0.02 = 2{,}112.03$$
$$PA_1 = Coup - InvInc_1 = 3{,}000 - 2{,}112.03 = 887.97$$
$$BV_1 = BV_0 - PA_1 = 105{,}601.43 - 887.97 = 104{,}713.46$$

The second full row of the amortization table is:

$$InvInc_2 = BV_1 \times y = 104{,}713.46 \times 0.02 = 2{,}094.27$$
$$PA_2 = Coup - InvInc_2 = 3{,}000 - 2{,}094.27 = 905.73$$
$$BV_2 = BV_1 - PA_2 = 104{,}713.46 - 905.73 = 103{,}807.73$$

The other rows are computed in a similar manner. The final row is:

$$InvInc_6 = BV_5 \times y = 100{,}980.39 \times 0.02 = 2{,}019.61$$
$$PA_6 = Coup - InvInc_6 = 3{,}000 - 2{,}019.61 = 980.39$$
$$BV_6 = BV_5 - PA_6 = 100{,}980.39 - 980.39 = 100{,}000$$

			Investment	Premium	End of Period
Year	Period	Coupon	Income	Amortized	Book Value
0.0	0				105,601.43
0.5	1	3,000.00	2,112.03	887.97	104,713.46
1.0	2	3,000.00	2,094.27	905.73	103,807.73
1.5	3	3,000.00	2,076.15	923.85	102,883.88
2.0	4	3,000.00	2,057.68	942.32	101,941.56
2.5	5	3,000.00	2,038.83	961.17	100,980.39
3.0	6	3,000.00	2,019.61	980.39	100,000.00

The BA II Plus can be used to find the amortization table:

6 [*N*]   2 [*I/Y*]   3,000 [*PMT*]   100,000 [*FV*]
[CPT] [*PV*]
Result is −105,601.43.   Price is 105,601.43.
[2nd] [AMORT] 1 [ENTER] ↓ 1 [ENTER] ↓

By continuing to hit the down arrow key we observe the values for the first full row:

BAL = −104,713.46
PRN = 887.97
INT = 2,112.03

The settings for P1 and P2 tell the calculator to determine the principal (i.e., premium amortized) and interest (i.e., investment income) included in payments P1 through P2. To find the principal and interest components of a single payment, we set P1 equal to P2.

The example below illustrates an amortization schedule for a discount bond.

Example 13.06

A 3-year bond makes semiannual coupon payments at an annual rate 6% per year. The par value is $100,000. The yield is 8% per year, compounded semiannually.

Prepare an amortization schedule for the bond.

Solution

Since the payments are made semiannually, the unit of time is 6 months:

$$c = \frac{0.06}{2} = 0.03 \qquad y = \frac{0.08}{2} = 0.08 \qquad n = 3 \times 2 = 6$$

The price of the bond is:

$$P = Coup \times a_{\overline{n|}y} + Rv^n = 0.03 \times 100{,}000 \times a_{\overline{6|}0.04} + \frac{100{,}000}{1.04^6}$$
$$= 3{,}000 \times \frac{1 - 1.04^{-6}}{0.04} + 79{,}039.45 = 3{,}000 \times 5.2421 + 79{,}039.45$$
$$= 94{,}757.86$$

Each row of the amortization table can be found with the following formulas:

$$InvInc_t = BV_{t-1} \times y$$

$$DA_t = InvInc_t - Coup$$

$$BV_t = BV_{t-1} + DA_t$$

The first full row of the amortization table is:

$$InvInc_1 = BV_0 \times y = 94,757.86 \times 0.04 = 3,790.31$$

$$DA_1 = InvInc_t - Coup = 3,790.31 - 3,000 = 790.31$$

$$BV_1 = BV_0 + DA_1 = 94,757.86 + 790.31 = 95,548.18$$

The other rows are computed in a similar manner. The final row is:

$$InvInc_6 = BV_5 \times y = 99,038.46 \times 0.04 = 3,961.54$$

$$DA_6 = InvInc_6 - Coup = 3,961.54 - 3,000 = 961.54$$

$$BV_6 = BV_5 + DA_6 = 99,038.46 + 961.54 = 100,000$$

Year	Period	Coupon	Investment Income	Discount Accrued	End of Period Book Value
0.0	0				94,757.86
0.5	1	3,000.00	3,790.31	790.31	95,548.18
1.0	2	3,000.00	3,821.93	821.93	96,370.10
1.5	3	3,000.00	3,854.80	854.80	97,224.91
2.0	4	3,000.00	3,889.00	889.00	98,113.91
2.5	5	3,000.00	3,924.56	924.56	99,038.46
3.0	6	3,000.00	3,961.54	961.54	100,000.00

The BA II Plus can be used to find the amortization table:

6 [*N*]   4 [*I/Y*]   3,000 [*PMT*]   100,000 [*FV*]

[CPT] [*PV*]

Result is −94,757.86.   Price is 94,757.86.

[2nd] [AMORT] 1 [ENTER] ↓ 1 [ENTER] ↓

By continuing to hit the down arrow key we observe the values for the first full row:

BAL = −95,548.18

PRN = −790.31

INT = 3,790.31

The PRN amount is the premium amortized. Since this is a discount bond, the premium amortized is negative, and it is equal to the opposite of the discount accrued.

13.03 Callable Bonds

Callable bonds are redeemable early at the option of the bond issuer on specified call dates at specified call prices. We refer to a call price at time *t* as the redemption value at time *t*.

The most common reason for a bond issuer to call a bond early is that the market interest rate at which the issuer can issue new bonds has fallen below the coupon rate on the callable bond, which allows the bond issuer to refinance at a lower interest rate.

Suppose that a callable bond is redeemable at *k* points in time for $R_{t_1}, R_{t_2}, \cdots, R_{t_k}$ at times $t_1, t_2, \cdots, t_k$ respectively. The bond matures at time *n*, so:

$$t_k = n \quad \text{and} \quad R_{t_k} = R = \text{Final redemption value}$$

For a given price *P*, there are *k* possible yields, depending on when the bond is redeemed:

$$y_{t_1}, y_{t_2}, \cdots, y_{t_k}$$

The lowest of these yields is called the **yield-to-worst**.

$$YTW = Min\left[y_{t_1}, y_{t_2}, \cdots, y_{t_k}\right] = \text{Yield to worst, given } P$$

If the bond is not redeemed until maturity, then its yield is known as its **yield-to-maturity**:

$$YTM = \text{Yield to maturity} = y_{t_k}$$

For a given yield y, there are k possible prices, depending on when the bond is redeemed. Each of these prices is a price to redemption (PTR):

$$PTR_{t_1}, PTR_{t_2}, \cdots, PTR_{t_k}$$

We call the minimum of these prices the **price-to-worst**, which is denoted by PTW:

$$PTW = Min\left[PTR_{t_1}, PTR_{t_2}, \cdots, PTR_{t_k}\right] = \text{Minimum price, given } y$$

If a YTW is used to calculate a PTW, then the resulting price is the price that was originally used to calculate the YTW. And if a PTW is used to calculate a YTW, then the resulting yield is the yield that was originally used to calculate the PTW.

Example 13.07

A 4-year bond makes annual coupon payments at an annual rate 10% per year. The par value is \$100. The annual effective yield is 6.50% per year.

The bond is callable after 1 year for \$110, after 2 years for \$106, and after 3 years for \$104. The redemption value after 4 years is \$100.

Calculate the price-to-worst.

Solution

Let's use the BA II Plus to obtain the 4 prices to redemption.

1 [*N*] 6.5 [*I/Y*] 10 [*PMT*] 110 [*FV*]
[CPT] [*PV*]
Result is −112.68. $PTR_1 = 112.68$

2 [*N*] 106 [*FV*]
[CPT] [*PV*]
Result is −111.66. $PTR_2 = 111.66$

3 [*N*] 104 [*FV*]
[CPT] [*PV*]
Result is −112.58. $PTR_3 = 112.58$

4 [*N*] 100 [*FV*]
[CPT] [*PV*]
Result is −111.99. $PTR_4 = 111.99$

The price-to-worst is:

$$PTW = Min\left[112.68, 111.66, 112.58, 111.99\right] = \mathbf{111.66}$$

In the next example, we use the same bond as in the example above, and we use the PTW to find the YTW.

Example 13.08

A 4-year bond makes annual coupon payments at an annual rate 10% per year. The par value is \$100. The price of the bond is \$111.66.

The bond is callable after 1 year for \$110, after 2 years for \$106, and after 3 years for \$104. The redemption value after 4 years is \$100.

Calculate the yield-to-worst.

Solution

Let's use the BA II Plus to obtain the 4 possible yields.

1 [*N*] −111.66 [*PV*] 10 [*PMT*] 110 [*FV*]

[CPT] [*I/Y*]

Result is 7.47. $y_1 = 7.47\%$

2 [*N*] 106 [*FV*]

[CPT] [*I/Y*]

Result is 6.50.

3 [*N*] 104 [*FV*]

[CPT] [*I/Y*]

Result is 6.82.

4 [*N*] 100 [*FV*]

[CPT] [*I/Y*]

Result is 6.59.

The yield-to-worst is:

$$YTW = Min\left[7.47\%, 6.50\%, 6.82\%, 6.59\%\right] = \mathbf{6.50\%}$$

As expected, using the *PTW* from the first example to obtain the *YTW* in the second example results in the same yield as the one provided in the first example.

If a bond purchaser pays more than *PTW*, then there is a possibility of earning a yield that is less than *y*.

Example 13.09

A 4-year bond makes annual coupon payments at an annual rate 10% per year. The par value is $100.

The bond is callable after 1 year for $110, after 2 years for $106, and after 3 years for $104. The redemption value after 4 years is $100.

The maximum price that can be paid for the bond to guarantee a yield of 6.50% is $111.66. An investor pays $112 for the bond because he observes that the yield-to-maturity is 6.50% at a price of $112.

Calculate the lowest possible yield that the investor could earn.

Solution

Let's use the BA II Plus to obtain the 4 possible yields.

1 [*N*] −112 [*PV*] 10 [*PMT*] 110 [*FV*]

[CPT] [*I/Y*]

Result is 7.14.

2 [*N*] 106 [*FV*]

[CPT] [*I/Y*]

Result is 6.33.

3 [*N*] 104 [*FV*]

[CPT] [*I/Y*]

Result is 6.70.

4 [*N*] 100 [*FV*]

[CPT] [*I/Y*]

Result is 6.50.

The yield-to-worst is:

$$YTW = Min\left[7.14\%, 6.33\%, 6.70\%, 6.50\%\right] = \mathbf{6.33\%}$$

If the investor in the example above needed a yield of 6.50%, then the investor needed to consider the possibility of early exercise.

To be assured of earning a yield of at least *y*, the bond purchaser must pay no more than the minimum of the purchase prices associated with that yield.

13.05

Maximum Price to Earn Desired Yield

The highest price that a purchaser can pay for a callable bond and be assured of earning a yield of at least y is the price–to-worst:

$$PTW = Min\left[PTR_{t_1}, PTR_{t_2}, \cdots, PTR_{t_k}\right] = \text{Minimum price, given } y$$

If we are asked for the maximum price that the purchaser can pay to obtain a desired yield, then, as indicated above, our task is to find the minimum of the prices that are calculated at that yield. The maximum that the purchaser can pay is the minimum of those prices. It can be confusing!

In this book, we usually subscript a price or a value with the time at which the value is found. For example, BV_t is the time-t book value. For prices that are found at various redemption dates, however, we deviate from that practice and use the subscript to indicate the time at which the bond is redeemed, so that PTR_{t_j} is the price of the bond at the outset, based on an assumption that it is redeemed at time t_j.

Example 13.10

A 4-year bond makes annual coupon payments at an annual rate 10% per year. The par value is $100.

The bond is callable after 1 year for $110, after 2 years for $106, and after 3 years for $104. The redemption value after 4 years is $100.

Calculate the maximum price that an investor can pay and still be assured of earning an annual effective yield of at least 12%.

Solution

Let's use the BA II Plus calculator to obtain the 4 prices to redemption.

1 [*N*] 12 [*I/Y*] 10 [*PMT*] 110 [*FV*]
[CPT] [*PV*]
Result is −107.14.
2 [*N*] 106 [*FV*]
[CPT] [*PV*]
Result is −101.40.
3 [*N*] 104 [*FV*]
[CPT] [*PV*]
Result is −98.04.
4 [*N*] 100 [*FV*]
[CPT] [*PV*]
Result is −93.93.

The price-to-worst is:

$$PTW = Min\left[107.14, 101.40, 98.04, 93.93\right] = \mathbf{93.93}$$

To find the *YTW* or the *PTW* for a bond, it isn't always necessary to consider all of the call dates. When the redemption prices monotonically decline, we may be able to skip calculating *YTW* or *PTW* values for some of the redemption dates.

Consider a callable bond that pays coupons of *Coup* and has k redemption values that are monotonically decreasing:

$$R_{t_1} \geq R_{t_2} \geq \cdots \geq R_{t_k}$$

Suppose that we are given either of the following facts about the bond:

- The coupon is less than the product of the yield-to-worst and the final redemption value.
- The price-to-worst is less than the final redemption value.

Being given either one of those facts implies the other, and since the price is less than the final redemption value, the bond is a discount bond:

Discount bond: $\quad Coup < YTW \times R_{t_k} \quad \Leftrightarrow \quad PTW < R_{t_k}$

Consider the perspective of someone who has just purchased the discount bond. The coupons are relatively low, so (assuming that the market interest rates have not fallen) the owner would prefer that the bond be paid off soon, allowing the owner to reinvest at a higher interest rate. The worst case is that the bond is not paid off until maturity, allowing the bond to pay its low coupon rate for an extended period. Therefore, both the yield-to-worst and the price-to-worst are found by assuming that the bond is not redeemed until maturity.

13.06

Discount Callable Bond

Consider a bond with monotonically declining redemption prices:

$$R_{t_1} \geq R_{t_2} \geq \cdots \geq R_{t_k}$$

If the bond satisfies one of the following equivalent conditions, then it is a discount bond:

$$Coup < YTW \times R_{t_k} \quad \Leftrightarrow \quad PTW < R_{t_k}$$

The price-to-worst (*PTW*) and the yield-to-worst (*YTW*) of the discount bond can be found by assuming that the bond is redeemed at maturity.

It might seem that the owner of a discount callable bond would always prefer to have the bond called as soon as possible because that would provide a high yield. But what if the bond issuer calls the bond because interest rates have fallen? In that event, even though the original bond's realized yield is increased by the early redemption, the lower reinvestment yields available in the bond market may leave the bond investor wishing that the original bond had not been called.

The example below is the same as the preceding example, except that the solution below uses the Key Concept above to answer the question more quickly.

Example 13.11

A 4-year bond makes annual coupon payments at an annual rate 10% per year. The par value is $100.

The bond is callable after 1 year for $110, after 2 years for $106, and after 3 years for $104. The redemption value after 4 years is $100.

Calculate the maximum price that an investor can pay and still be assured of earning an annual effective yield of at least 12%.

Solution

The coupon payment is less than the product of the yield-to-worst and the final redemption value, so the bond is a discount bond:

$$0.10 \times 100 < 0.12 \times 100$$
$$10 < 12$$

Since the redemption prices are monotonically decreasing, the worst yield is realized if the bond is held until maturity.

We use the BA II Plus calculator to obtain the price if the bond is held to maturity.

4 [*N*] 12 [*I/Y*] 10 [*PMT*] 100 [*FV*]

[CPT] [*PV*]

Result is −93.93. $PTR_4 = 93.93$

The price-to-worst is 93.33, which means that the maximum price that an investor can pay and still be assured of earning an annual effective yield of at least 12% is **93.33**.

Consider again a callable bond that pays coupons of *Coup* and has *k* redemption values that are monotonically decreasing:

$$R_{t_1} \geq R_{t_2} \geq \cdots \geq R_{t_k}$$

Now suppose that we are given either of the following facts about the bond:

- The coupon is greater than the product of the yield-to-worst and the final redemption value.
- The price-to-worst is greater than the final redemption value.

Being given either one of those facts implies the other, and since the price is greater than the final redemption value, the bond is a premium bond:

$$\text{Premium bond:} \qquad Coup > YTW \times R_{t_k} \qquad \Leftrightarrow \qquad PTW > R_{t_k}$$

Consider the perspective of someone who has just purchased the premium bond. The coupons are relatively high, so (assuming that the market interest rates have not risen) the owner would prefer that the bond be redeemed as far into the future as possible, allowing the owner to collect the high coupons for a longer time. On the other hand, since the call price declines over time, an early call brings a higher call price. From the bond purchaser's perspective, an early call has two offsetting impacts:

− The high coupons are received for a shorter period of time.

+ The redemption value may be higher.

Over a time interval during which the redemption value is constant, the worst case within that interval is that the bond is called at the earliest opportunity. Therefore, both the yield-to-worst and the price-to-worst within that interval are found by assuming that the bond is redeemed at the beginning of the interval.

13.07

Premium Callable Bond

Consider a bond with monotonically declining redemption prices:

$$R_{t_1} \geq R_{t_2} \geq \cdots \geq R_{t_k}$$

If the bond satisfies one of the following equivalent conditions, then it is a premium bond:

$$Coup > YTW \times R_{t_k} \qquad \Leftrightarrow \qquad PTW > R_{t_k}$$

The price-to-worst (*PTW*) and the yield-to-worst (*YTW*) of the premium bond can be found by identifying each interval defined by level redemption prices. The worst prices/yields within those intervals are found at the beginning of those intervals.

The example below has 13 possible redemption dates, but the Key Concept above tells us that we need only consider 4 of those redemption dates to find the *PTW*.

Example 13.12 A 10-year bond makes semiannual coupon payments at an annual rate 10% per year. The par value is \$100.

The bond is callable after 4 years for \$110, after 6 years for \$108, and after 8 years for \$103. The redemption value upon maturity is \$100.

Calculate the maximum price that an investor can pay and still be assured of earning a yield of at least 8% compounded semiannually.

Solution The bond is callable on any of the payment dates on or after 4 years. This means that the bond could be redeemed at any of the following 13 times:

4, 4.5, 5, 5.5, 6, 6.5, 7, 7.5, 8, 8.5, 9, 9.5, and 10 years

The coupon payment is greater than the product of the yield-to-worst and the final redemption value, so the bond is a premium bond:

$$\frac{0.10}{2} \times 100 > 0.04 \times 100$$

$$5 > 4$$

Since the redemption prices are monotonically decreasing, we only check the call prices at the beginning of each period to which they first apply.

This means that the redemption times to check are:

4, 6, 8, and 10 years

Since the bond makes semiannual payments, we use 6 months as our unit of time. Therefore, the redemption times to be considered, measured in 6-month periods, are:

8, 12, 16, and 20

Let's use the BA II Plus calculator to obtain the 4 prices to redemption.

8 [*N*]   4 [*I/Y*]   5 [*PMT*]   110 [*FV*]
[CPT] [*PV*]
Result is −114.04. $PTR_8 = 114.04$

12 [*N*]   108 [*FV*]
[CPT] [*PV*]
Result is −114.38. $PTR_{12} = 114.38$

16 [*N*]   103 [*FV*]
[CPT] [*PV*]
Result is −113.25. $PTR_{16} = 113.25$

20 [*N*]   100 [*FV*]
[CPT] [*PV*]
Result is −113.59. $PTR_{20} = 113.59$

The price-to-worst is the price that is based on the bond being called after 8 years:

$$PTW = Min\left[114.04, 114.38, 113.25, 113.59\right] = \mathbf{113.25}$$

13.04 Government Securities

The debt issued by the United States Treasury is often referred to as Treasuries, and it includes the following:

- Treasury bills – Issued with terms of 1 year or less
- Notes – Issued with terms of 2 to 10 years
- Bonds – Issued with terms of more than 10 years
- TIPS – Treasury Inflation-Protected Securities

The annualized yields on these securities as of 7/3/2018 are shown below:

Maturity	Bills	Notes/Bonds	TIPS
1 month	1.88%	1.91%	
3 months	1.94%	1.98%	
6 months	2.07%	2.12%	
1 year	2.25%	2.33%	
2 year		2.53%	
3 year		2.63%	
5 year		2.72%	0.64%
7 year		2.79%	0.68%
10 year		2.83%	0.71%
20 year		2.89%	0.78%
30 year		2.96%	0.83%

Treasury bills are sold at a discount and do not pay coupons. Treasury bills, which are also known as T-bills, are discussed in more detail in Section 13.06. The Treasury bill

yields in the table above don't quite match the yields in the Notes/Bonds column because the calculation basis for the yields is not the same. The calculation basis for T-bills is discussed in Section 13.06. The yields shown in the Notes/Bonds and TIPS columns are expressed as interest rates that are compounded twice a year because those securities make coupon payments semiannually.

Although the Treasury Department refers to its intermediate-term debt as "notes", traders and investors usually refer to them as "bonds." In general, the term "bond" is used to describe debt that is issued with a term of 1 or more years. The table above contains yields on notes and bonds with maturities of less than 1 year because as the securities age, their maturities decline, so outstanding notes and bonds can mature within one year.

U.S. government securities are often described as being risk-free because the government's ability to create money ensures that the bonds will be paid off in full. Some risk remains, however:

- Price risk – If interest rates rise, then the value of the bond may fall prior to maturity.
- Inflation risk – The purchasing power of the securities' payments could be eroded by inflation.
- Currency risk – An investor whose liabilities are not denominated in dollars faces the risk that the dollar could depreciate relative to the investor's primary currency.

Treasury notes and Treasury bonds are sometimes called **nominal return bonds** to differentiate them from securities that have inflation protection. Treasury Inflation-Protected Securities (TIPS) are known as **real return bonds** or inflation-indexed bonds because the interest and principal payments of TIPS are indexed with inflation. TIPS are a useful tool for investors who are concerned about inflation risk, such as pension plans and retirees. To compensate for the inflation protection provided by TIPS, their yields are less than the yields of nominal return bonds. In the table above, the yields shown in the TIPS column are the real rates of return. The actual rate of return earned on the TIPS is the real rate of return plus the inflation rate. The yields on TIPS have been negative in the past, indicating that the real rate of return on those instruments was negative.

States, cities, and other units of local government can also issue bonds, which are called **municipal bonds** or "munis." These bonds are classified according to the source of the revenue that is used to retire the bonds:

- **General obligation bonds** rely on the taxing authority of the issuer.
- **Revenue bonds** rely on the funds generated by a project or specified source, such as the tolls paid by drivers to use a bridge.

Since the issuers of municipal bonds are not able to create money, these bonds can default. Despite the default risk, many munis have yields that are less than the yields on Treasuries because the interest income from qualifying municipal bonds is exempt from federal tax and is sometime free of local income taxes as well. The tax benefit of owning the municipal benefit can outweigh the risk of default.

Like the United States, Canada issues short-term and long-term debt. As in the United States, the short-term debt is called bills. More information about Canadian Treasury bills appears in Section 13.07. The long-term debt is referred to as bonds, and it is primarily denominated in Canadian dollars. Within Canada, this debt is considered to be risk-free, although it is subject to the same kinds of price risk, inflation risk, and currency risk described above for U.S. Treasuries. To eliminate the currency risk for U.S. investors, Canada issues some debt that is denominated in U.S. dollars. Although this eliminates the currency risk for U.S. investors, it raises the possibility of default, since the Canadian government is not able to create U.S. dollars. Since the Canadian government is financially stable, the risk of default is considered to be quite low. Canadian government bonds denominated in dollars trade at slightly higher yields than U.S. Treasuries. The primary reasons are the small possibility of default and the fact that the market for Canadian government bonds is not as liquid as the market for U.S. Treasuries.

Canadian provinces and local units of government also issue debt, but, unlike U.S. munis, this debt is not tax-exempt. Since these bonds could default, the yields on these bonds are higher than the yields on comparable Canadian government bonds.

13.05 Corporate Bonds

Corporate bonds are bonds that are issued by corporations. If a corporation issues a bond and is subsequently unable to make the coupon and principal payments, then the bond is said to be in default. Rating agencies, such as Moody's and Standard & Poor's, provide credit ratings that indicate the likelihood that a bond will go into default. The table below provides the credit ratings used by the two agencies, from highest to lowest.

Moody's	**S&P**
Aaa	AAA
Aa1	AA+
Aa2	AA
Aa3	AA-
A1	A+
A2	A
A3	A-
Baa1	BBB–
Baa2	BBB
Baa3	BBB–
Ba1	BB+
Ba2	BB
Ba3	BB–
B1	B+
B2	B
B3	B–
Caa1	CCC+
Caa2	CCC
Caa3	CCC–
Ca	CC
C	C

The horizontal line in the table above indicates the cutoff for investment-grade bonds. Bonds with ratings below the line are not investment grade, and they are called high-yield bonds or junk bonds.

In the event that a bond goes into default, the bondholder does not receive the full timely payment of the coupons and principal, but a partial recovery is possible and even likely, depending on the seniority of the bond. Bonds with higher seniority are paid off before bonds with lower seniority. Lower seniority bonds are known as subordinated bonds, and they generally have higher yields than the more senior bonds to compensate the bond purchaser for the greater risk of default losses. Full or partial recovery may also be possible if the bond is secured, or collateralized, with specific assets that the corporation has pledged for the fulfillment of a bond's obligations.

Investment brokers maintain bond inventories and provide ask prices and bid prices to their clients. The ask price is the price at which an investor can purchase the bond from the broker, and the (lower) bid price is the price at which the broker is willing to buy the bond from an investor. If the bid price were to be greater than the ask price, then the broker's clients could earn arbitrage profits, and the broker would incur losses. Since no broker wishes to incur losses, the ask price is greater than the bid price. The broker may quote a yield instead of a price, and the inverse relationship between yields and prices implies that the **ask yield** is lower than the **bid yield**:

$$\text{ask price} > \text{bid price} \quad \Leftrightarrow \quad \text{ask yield} < \text{bid yield}$$

When thinking through the relationship between the bid price and the ask price, it can be helpful to keep in mind that it is the broker, not the client, that does the bidding and the asking.

The difference between the bid yield and the ask yield is the **bid-ask spread**. The bid-ask spread is an indication of the **liquidity** of a bond. A liquid asset is an asset that can be converted to cash with relatively little loss in value. A bond with a lower bid-ask spread is more liquid than a bond with a higher bid-ask spread.

As we saw in Section 13.03, callable bonds give the bond issuer the right to redeem the bond early. A **puttable bond** provides a similar right to the bond owner. The owner of a puttable bond can sell, or put, the bond to the bond issuer at a specified redemption price prior to the maturity date of the bond. The redemption prices of callable bonds are usually greater than or equal to par, and the redemption prices of puttable bonds are usually less than or equal to par. To compensate for the risk that a callable bond will be called, the yield of a callable bond is higher than the yield of a regular bond. A puttable bond has a put provision that gives the bond's owner an additional right, and the owner of a puttable bond pays for this right by accepting a yield that is lower than the yield of regular bond:

$$\text{Puttable bond yield} < \text{Regular bond yield} < \text{Callable bond yield}$$

The price of a corporate bond varies over time, and therefore its yield changes as well. The main factors affecting the yield of a corporate bond include the following:

- Treasury yields of comparable term Treasuries
- Default risk
- Seniority
- Liquidity
- Call and put features
- Currency in which the bond is denominated
- Remaining time until maturity
- Inflation risk

13.06 U.S. Treasury Bills

Treasury bills, or T-bills, do not pay coupons. Instead, a T-bill is purchased at a discount and the purchaser receives the maturity value when the T-bill matures. The interest earned on a T-bill is the difference between the maturity value and the price:

$$Interest = MaturityValue - Price$$

The quoting convention for a U.S. T-bill makes use of a simple discount rate based on a 360-day year. Recall from Section 2.02 that a simple discount rate can be used to find the present value of an accumulated value as follows:

$$PV_0 = AV_t(1 - dt)$$

Let's refer to the present value as the price, the accumulated value as the maturity value, and the discount rate as the quoted rate. Furthermore, let's define the time increment to be the actual number of days until maturity divided by 360. The formula for the price of a U.S. Treasury bill can now be determined.

U.S. Treasury Bill Price

13.08 In order to calculate the price of a U.S. Treasury bill, the quoted rate is treated as a simple discount rate, and the time interval is calculated on an actual/360 basis:

$$Price = MaturityValue\left[1 - QR_{US} \times \frac{DaysToMaturity}{360}\right]$$

where:

$$QR_{US} = \text{Quoted rate on a U.S. Treasury bill}$$

In the next example, a quoted rate is used to find the price of a U.S. T-bill.

Example 13.13 The quoted rate of a U.S. Treasury bill is 5.70%. The Treasury bill matures for 1,000,000 in 58 days. Calculate the price of the Treasury bill.

Solution The price is the present value of the maturity value based on a simple discount rate of 5.70%:

$$Price = MaturityValue\left[1 - QR_{US} \times \frac{DaysToMaturity}{360}\right]$$

$$= 1{,}000{,}000\left[1 - 0.057 \times \frac{58}{360}\right] = \mathbf{990{,}816.67}$$

Earlier, we observed that the interest earned on a T-bill is the maturity value minus the price:

$$Interest = MaturityValue - Price$$

Substituting in for the price, we see that the interest can be expressed as the quoted rate times the maturity value times the amount of time until maturity:

$$\begin{aligned} Interest &= MaturityValue - Price \\ &= MaturityValue - MaturityValue\left[1 - QR_{US} \times \frac{DaysToMaturity}{360}\right] \\ &= QR_{US} \times MaturityValue \times \frac{DaysToMaturity}{360} \end{aligned}$$

Rearranging, we have a formula for the quoted rate:

$$Interest = QR_{US} \times MaturityValue \times \frac{DaysToMaturity}{360}$$

$$QR_{US} = \frac{360}{DaysToMaturity} \times \frac{Interest}{MaturityValue}$$

13.09

U.S. Treasury Bill Quoted Rate

The quoted rate on a U.S. Treasury bill is an annualized discount rate based on a 360-day year:

$$QR_{US} = \frac{360}{DaysToMaturity} \times \frac{Interest}{MaturityValue}$$

In the Key Concept above, the interest is divided by the maturity value rather than the price because the quoted rate is a discount rate, not an interest rate. The resulting rate is annualized by multiplying by 360 and dividing by the days until maturity. For example, if there are 30 days until maturity, then the rate is annualized by multiplying by 360/30 = 12.

A discount rate is always less than its equivalent interest rate. Therefore, as shown in the example below, the quoted rate is less than the annual effective yield.

Example 13.14 The price of a U.S. Treasury bill is 96,500. The Treasury bill matures for 100,000 in 280 days.

a. Calculate the quoted rate of the Treasury bill.
b. Calculate the annual effective yield of the Treasury bill.

Solution a. The interest is the difference between the maturity value and the price:

$$Interest = MaturityValue - Price = 100{,}000 - 96{,}500 = 3{,}500$$

The quoted rate is an annualized discount rate:

$$QR_{US} = \frac{360}{DaysToMaturity} \times \frac{Interest}{MaturityValue} = \frac{360}{280} \times \frac{3,500}{100,000} = \mathbf{4.50\%}$$

b. The annual effective yield, y, is the annual effective interest rate at which 96,500 would grow to 100,000:

$$96,500(1+y)^{280/365} = 100,000$$

$$y = \mathbf{4.75\%}$$

Unless we are told otherwise, we should assume the following:

- *a yield is an interest rate, and*
- *an annual effective interest rate is based on a 365-day year.*

13.07 Canadian Treasury Bills

The quoting convention for a Canadian T-bill makes use of a simple interest rate based on a 365-day year. Recall from Section 2.01 that a simple interest rate can be used to find the present value of an accumulated value as follows:

$$PV_0 = \frac{AV_t}{(1+it)}$$

Let's refer to the present value as the price, the accumulated value as the maturity value, and the interest rate as the quoted rate. Furthermore, let's define the time increment to be the actual number of days until maturity divided by 365. The formula for the price of a Canadian Treasury bill can now be determined.

Canadian Treasury Bill Price

13.10 In order to calculate the price of a Canadian Treasury bill, the quoted rate is treated as a simple interest rate, and the time interval is calculated on an actual/365 basis:

$$Price = \frac{MaturityValue}{1 + QR_C \times \dfrac{DaysToMaturity}{365}}$$

where:

$$QR_C = \text{Quoted rate on a Canadian Treasury bill}$$

In the next example, the quoted rate is used to find the price of a Canadian Treasury bill.

Example 13.15

The quoted rate of a Canadian Treasury bill is 5.70%. The Treasury bill matures for 1,000,000 in 58 days. Calculate the price of the Treasury bill.

Solution

The price is the present value of the maturity value based on a simple interest rate of 5.70%:

$$Price = \frac{MaturityValue}{1 + QR_C \times \dfrac{DaysToMaturity}{365}} = \frac{1,000,000}{1 + 0.057 \times \dfrac{58}{365}} = \frac{1,000,000}{1.0090575} = \mathbf{991,023.77}$$

As is the case with U.S. T-bills, the interest earned on a Canadian T-bill is the maturity value minus the price:

$$Interest = MaturityValue - Price$$

Substituting in for the price, we see that the interest is the quoted rate times the price times the amount of time until maturity:

$$\begin{aligned} Interest &= MaturityValue - Price \\ &= Price \times \left[1 + QR_C \times \frac{DaysToMaturity}{365}\right] - Price \\ &= QR_C \times Price \times \frac{DaysToMaturity}{365} \end{aligned}$$

Rearranging, we have a formula for the quoted rate:

$$Interest = QR_C \times Price \times \frac{DaysToMaturity}{365}$$

$$QR_C = \frac{365}{DaysToMaturity} \times \frac{Interest}{MaturityValue}$$

13.11

Canadian Treasury Bill Quoted Rate

The quoted rate on Canadian Treasury bill is an annualized interest rate based on a 365-day year:

$$QR_C = \frac{365}{DaysToMaturity} \times \frac{Interest}{Price}$$

In the Key Concept above, the interest is divided by the price rather than the maturity value because the quoted rate is an interest rate, not a discount rate.

The quoted rate on a U.S. T-bill is a discount rate while the quoted rate of a Canadian T-bill is an interest rate. A discount rate is always less than its equivalent interest rate. This implies that if a U.S. T-bill and a Canadian T-bill have the same price, maturity value, and time until maturity, then the quoted rate of the U.S. T-bill is less than the quoted rate of the Canadian T-bill. Another way to see this is to compare the formulas for the quoted rate:

$$QR_{US} = \frac{360}{DaysToMaturity} \times \frac{Interest}{MaturityValue} \quad \& \quad QR_C = \frac{365}{DaysToMaturity} \times \frac{Interest}{Price}$$

The fact that the maturity value is greater than the price is sufficient to show that the U.S. quoted rate is less than the Canadian quoted rate. Furthermore, the U.S. quoted rate is based on a 360-day year instead of a 365-day year, which also has the effect of making the U.S. quoted rate smaller than the Canadian quoted rate:

$$MaturityValue > Price \quad \& \quad 360 < 365 \quad \Rightarrow \quad QR_{US} < QR_C$$

Example 13.16

Two Treasury bills each mature for 400,000 in 32 days, and both bills have a current price of 398,000.

One of the Treasury bills is a U.S. Treasury bill, and its maturity value and price are denominated in U.S. dollars.

The other Treasury bill is a Canadian Treasury bill, and its maturity value and price are denominated in Canadian dollars.

Calculate the amount by which the quoted rate on the Canadian Treasury bill exceeds the quoted rate on the U.S. Treasury bill.

Solution

The quoted rates are:

$$\begin{aligned} QR_{US} &= \frac{360}{DaysToMaturity} \times \frac{Interest}{MaturityValue} = \frac{360}{32} \times \frac{400{,}000 - 398{,}000}{400{,}000} \\ &= 5.6250\% \end{aligned}$$

$$QR_C = \frac{365}{DaysToMaturity} \times \frac{Interest}{Price} = \frac{365}{32} \times \frac{400{,}000 - 398{,}000}{398{,}000} = 5.7318\%$$

The difference is:

$$QR_C - QR_{US} = 5.7318\% - 5.6250\% = \mathbf{0.1068\%}$$

13.08 Accrued Interest

Suppose that at time t, a noncallable bond has n coupon payments remaining, and the next coupon payment occurs in one unit of time. Then the bond prices at time t and time $(t+1)$ are:

$$P_t = Coup \times a_{\overline{n|}y} + Rv^n \qquad \text{where: } v = \frac{1}{1+y}$$

$$P_{t+1} = Coup \times a_{\overline{n-1|}y} + Rv^{n-1}$$

Now suppose that the bond is purchased between coupon payments, at time $(t+h)$, where $0 < h < 1$ and:

$$h = \frac{\text{Number of days from last coupon payment to settlement date}}{\text{Number of days between coupon payments}}$$

If the bond is a recently issued bond, then it may not yet have paid any coupons as of the settlement date, in which case the numerator of h becomes the number of days from the issue date to the settlement date.

The **full price** of a bond is the price that is paid for the bond when it is purchased. The full price is sometimes called the price-plus-accrued or the dirty price. The time of purchase is called the **settlement date**.

The trade date, which comes a few days before the settlement date, is the date that the terms of the purchase are agreed upon. We use the settlement date, not the trade date, when calculating h.

The full price of the bond on the settlement date is equal to the present value of the future cash flows:

$$FP_{t+h} = Coup\left(v^{1-h} + v^{2-h} + \cdots + v^{n-h}\right) \times +Rv^{n-h}$$

$$FP_{t+h} = v^{-h}\left[Coup\left(v + v^2 + \cdots + v^n\right) \times +Rv^n\right]$$

$$FP_{t+h} = (1+y)^h\left[Coup \times a_{\overline{n|}y} + Rv^n\right]$$

$$FP_{t+h} = (1+y)^h \times P_t$$

The final expression above describes the full price at time $(t+h)$ as the accumulated value of the time t price.

Alternatively, we can find the price as the discounted value of the time $(t+1)$ price plus the present value of the coupon paid at time $(t+1)$:

$$FP_{t+h} = Coup\left(v^{1-h} + v^{2-h} + \cdots + v^{n-h}\right) \times +Rv^{n-h}$$

$$FP_{t+h} = v^{1-h}\left[Coup\left(1 + v + \cdots + v^{n-1}\right) \times +Rv^{n-1}\right]$$

$$FP_{t+h} = v^{1-h}\left[Coup + Coup\left(v + \cdots + v^{n-1}\right) \times +Rv^{n-1}\right]$$

$$FP_{t+h} = v^{1-h}\left[Coup + Coup \times a_{\overline{n-1|}y} + Rv^{n-1}\right]$$

$$FP_{t+h} = v^{1-h}\left[Coup + P_{t+1}\right]$$

The full price can also be written as the sum of the **quoted price** (QP) and the **accrued interest** (AI). The quoted price is also known as the **clean price**, the flat price, or the market price:

$$FP_{t+h} = QP_{t+h} + AI$$

where:

$$AI = h \times Coup$$

As shown above, the accrued interest could be more accurately described as the **accrued coupon**. The term accrued interest is commonly used in the bond market though, so we use it here as well. The accrued interest is the pro rata portion of the coupon that is earned since the last coupon payment date, based on a simple interest calculation.

The quoted price is useful for monitoring price movements in the bond that are due to factors other than the passage of time since the last coupon payment. If the yield of a bond does not change over time, then its quoted price doesn't change much either.

When a bond is purchased immediately after a coupon payment, its full price and its quoted price are equal.

Since there are many terms to describe the same things, let's summarize:

Full price = Price-plus-accrued = Dirty price

Quoted price = Clean price = Flat price = Market price

Accrued interest = Accrued coupon

13.12

Full Price and Quoted Price

Suppose that a bond is purchased at time $(t+h)$, at which time there are n coupon payments remaining, and:

$$h = \frac{\text{Number of days from last coupon payment to settlement date}}{\text{Number of days between coupon payments}}$$

The full price of the bond can be found retrospectively or prospectively:

$$FP_{t+h} = (1+y)^h \times P_t$$

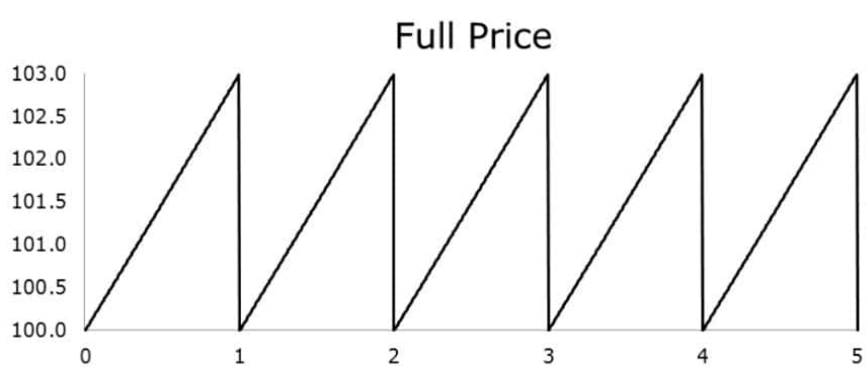
$$FP_{t+h} = v^{1-h}\left[Coup + P_{t+1}\right]$$

The accrued interest and the quoted price are:

$$AI = h \times Coup$$

$$QP_{t+h} = FP_{t+h} - AI$$

The full price falls immediately after a coupon payment is made. The graph below shows the full price throughout the life of a 5-year bond that pays annual coupons of 3% per year, based on an annual effective yield of 3%. Immediately after each coupon is paid, the full price falls to the par value of $100.



Day Counts

The calculation of h above is based on the number of days in both the numerator and the denominator. The two most common ways to count days are referred to as **actual/actual** and **30/360**.

Actual/actual is usually used for government bonds, and as the name implies, the actual number of days from the last coupon payment date (or issue date) to the settlement date is used in the numerator, and the actual number of days between coupon payments is used in the denominator.

The 30/360 method is usually applied for corporate bonds. Using this method, months are assumed to have 30 days each, and each year is assumed to have 360 days.

The date worksheet included with the BA II Plus calculator makes it easy to count the number of days between two dates using either method. And the bond worksheet automatically counts the days and computes the flat price and the accrued interest.

Example 13.17

A bond with a par value of $100 makes semiannual coupon payments at an annual rate 8% per year. The bond makes coupon payments on June 1 and December 1 of each year, and the bond matures on December 1, 2025.

The bond was purchased on September 10, 2015 to yield 6% per year compounded semiannually.

a. Calculate the number of days from the last coupon date to the settlement date. Use the 30/360 day count method.
b. Calculate h.
c. Calculate the full price.
d. Calculate the accrued interest.
e. Calculate the quoted price.

Solution

a. We need to find the number of days between 6/1/2015 and 9/10/2015. There are 29 days remaining in June after June 1. We treat July and August as if they have 30 days each. The first 10 days of September are fully counted, since 10 does not exceed 30:

$$29 + 30 + 30 + 10 = \mathbf{99}$$

Alternatively, we can use the BA II Plus calculator to count the days:

[2nd] [DATE]
6.0115 [ENTER] ↓
9.1015 [ENTER] ↓ ↓
(Use [2nd] [SET] if necessary so that 360 appears)
↑ [CPT]

Days between dates = **99**

b. Under the 30/360 method, the number of days between semiannual coupon payments is 180:

$$h = \frac{\text{Number of days from last coupon payment to settlement date}}{\text{Number of days between coupon payments}}$$

$$= \frac{99}{180} = \mathbf{0.55}$$

c. The bond pays 1 coupon in 2015, and 2 coupons in each following year up to and including 2025. Therefore, there is one year in which the bond pays 1 coupon and 10 years in which it pays 2 coupons. The number of remaining coupon payment periods is:

$$1 + 10 \times 2 = 21$$

On June 1, 2015, the price of the bond would be:

$$P_t = 4 \times a_{\overline{21}|0.03} + \frac{100}{1.03^{21}} = 4 \times \frac{1 - 1.03^{-21}}{0.03} + 53.7549$$

$$= 4 \times 15.4150 = 115.4150$$

The full price of the bond on September 10, 2015 is:

$$FP_{t+h} = (1+y)^h \times P_t = (1.03)^{0.55} \times 115.4150 = \mathbf{117.3067}$$

d. The accrued interest is:

$$AI = h \times Coup = 0.55 \times 4 = \mathbf{2.20}$$

e. The quoted price is:

$$QP_{t+h} = FP_{t+h} - AI = 117.3067 - 2.20 = \mathbf{115.1067}$$

Alternatively, we can answer parts c, d, and e in the example above using the bond worksheet:

[2nd] [BOND] 9.1015 [ENTER] ↓

8 [ENTER] ↓

12.0125 [ENTER] ↓

100 [ENTER] ↓

(Use [2nd] [SET] as necessary so that 360 appears) ↓

(Use [2nd] [SET] as necessary so that 2/Y appears) ↓

6 [ENTER] ↓

[CPT]

115.1067 is the result. This is the quoted price.

↓

2.20 is the result. This is the accrued interest.

The full price is obtained by adding the quoted price and the accrued interest:

$115.1067 + 2.20 = \mathbf{117.3067}$

The BA II Plus calculator can be a real time saver for these kinds of questions!

13.09 Questions

Question 13.01

A ten-year bond with a par value of 100 pays semiannual coupons at an annual rate of 7%. The bond is priced at 129.24 to yield a nominal annual rate of 4% convertible semiannually.

Calculate the redemption value of the bond.

A 72.01 B 76.26 C 107.00 D 112.98 E 119.22

Question 13.02

A 25-year bond with a par value of 1,000 and 10% coupons payable quarterly is selling at 925.

Calculate the annual nominal yield rate convertible quarterly.

A 2.7% B 5.4% C 8.2% D 10.9% E 13.6%

Question 13.03

A zero-coupon bond that will pay 1,000 at the end of 12 years is selling for 556.84.

A 5% annual coupon bond with coupons payable semiannually will pay 1,000 at the end of 15 years. The price of the 5% coupon bond is X, and it has the same annual effective yield as the zero-coupon bond.

Calculate X.

A 1,000.00 B 1,006.41 C 1,008.24 D 1,010.59 E 1,012.17

Question 13.04

A 1,000 par value 30-year bond sells for X and yields a nominal interest rate of 9% convertible semiannually. The bond has 8% coupons payable semiannually and a redemption value of 1,150.

Calculate X.

A 896.81 B 900.39 C 903.94 D 907.50 E 1,000.00

Question 13.05

An n-year bond has the following characteristics:

Coupons are paid semiannually.

Par value is 1,000.

The ratio of the semiannual coupon rate r to the semiannual yield rate i is 1.25.

The present value of the redemption value is 543.

Given that $\left(\frac{1}{1+i}\right)^n = 0.7026$, calculate the price of the bond.

A 914.75 B 1,000.00 C 1,080.03 D 1,160.06 E 1,175.94

Question 13.06

An n-year bond has the following characteristics:

Coupons are paid semiannually.

Par value is 1,000.

The ratio of the semiannual coupon rate r to the semiannual yield rate i is 1.25.

The price of the bond is 1,225.

Given that $\left(\frac{1}{1+i}\right)^n = 0.7026$, calculate the redemption value of the bond.

A 1,199.36 B 1,204.38 C 1,209.40 D 1,214.42 E 1,225.00

Question 13.07

Two 20-year bonds have the same purchase price. Both bonds have a par value of 1,000, and each has an annual coupon rate of 6% paid semiannually.

The first bond has an annual nominal yield rate of 6% compounded semiannually, and its redemption value is 1,100.

The second bond has an annual nominal yield rate of i compounded semiannually, and its redemption value is 912.

Calculate i.

A 2.7% B 4.9% C 5.5% D 6.0% E 6.6%

Question 13.08

Two 20-year bonds have the same purchase price. Both bonds have the same par value of X, and each has an annual coupon rate of 6% paid semiannually.

The first bond has an annual nominal yield rate of 6% compounded semiannually, and its redemption value is $1.1X$.

The second bond has an annual nominal yield rate of i compounded semiannually, and its redemption value is $0.895X$.

Calculate i.

A 2.7% B 3.4% C 4.1% D 4.8% E 5.5%

Question 13.09

Sally purchases a 20-year bond with semiannual coupons, redeemable at par, for a price of 100,000. The annual effective yield rate is 5%, and the annual coupon rate is 4.95%.

Calculate the redemption value of the bond.

A 99,862 B 100,000 C 100,238 D 100,476 E 100,627

Question 13.10

An investor purchases a 1,000 par value 10-year bond with 7% coupons that are paid semiannually. The yield on the bond is 5% convertible semiannually.

The coupon payments from the bond are reinvested at a nominal rate of 3% convertible semiannually.

The investor borrows the amount necessary to purchase the bond and will repay all interest and principal in a lump sum at the end of 10 years. The annual effective interest rate on the loan is 4%.

Calculate the net gain to the investor at the end of 10 years after the loan is repaid.

A 0.00 B 98.33 C 196.67 D 555.11 E 653.44

Question 13.11

You are given the following information about Bond X and Bond Y:

Both bonds are 10-year bonds with semiannual coupons, redeemable at their par value of 10,000, and both are bought to yield an annual nominal interest rate of i, convertible semiannually.

Bond X has an annual coupon rate of $(i + 0.02)$, paid semiannually.

Bond Y has an annual coupon rate of $(i - 0.02)$, paid semiannually.

The price of Bond X is 2,816.93 greater than the price of Bond Y.

Calculate i.

A 3.6% B 4.8% C 6.0% D 7.2% E 8.4%

Question 13.12

Steve has 7,000 and he borrows 3,000, so that he can pay 10,000 for a 15-year bond with a par value of 10,000 and 10% coupons paid monthly.

The loan for 3,000 is a 15-year, interest-only loan, requiring that he make interest payments at the end of each month at a nominal rate of 8% convertible monthly. At the end of 15 years, he repays the loan in full.

Calculate the annual effective yield that Steve will realize on his 7,000 over the 15-year period.

A 10.5% B 11.4% C 12.7% D 14.0% E 15.2%

Question 13.13

Sally owns a bond that is redeemable for 300 in 7 years. Sally has just received a coupon of c and each subsequent semiannual coupon will be 3% larger than the preceding coupon. The present value of this bond immediately after the payment of the coupon is 708.11, based on an annual effective yield of 5%.

Calculate c.

A 31.2 B 32.6 C 34.0 D 35.4 E 36.8

Question 13.14

An investor buys a 12-year 1,000 par value bond with semiannual coupons paid at an annual rate of 8%. The nominal yield on the bond is 8%, convertible semiannually.

As the investor receives each coupon payment, he immediately deposits the money into an account earning interest at an annual effective rate of i.

At the end of 12 years, immediately after receiving the final coupon payment and the redemption value of the bond, the investor has earned an annual effective yield of 9% on his investment.

Calculate i.

A 5.1% B 6.5% C 7.8% D 9.2% E 10.5%

Question 13.15

Three annual coupon bonds are redeemable at par, and all three have the same term, price and yield.

The first bond has a face value of 1,200 and an annual coupon rate of 5.17%.

The second bond has a face value of 1,050 and an annual coupon rate of 6.48%.

The third bond has a face value of 950 and an annual coupon rate of r.

Calculate r.

A 7.6% B 8.0% C 8.6% D 9.2% E 9.8%

Question 13.16

Christina purchases a 1,000 par value 15-year bond with 7% annual coupons that are paid semiannually. She pays 1,245 for the bond. She reinvests her coupon payments at a nominal rate of 5% convertible semiannually.

Calculate the nominal yield convertible semiannually that she earns over the 15-year interval.

A 2.4% B 3.0% C 3.6% D 4.2% E 4.8%

Question 13.17

A 1,000 par value 15-year bond with 6% annual coupons is bought at a premium to yield an annual effective rate of 4%.

Calculate the interest portion of the 8^{th} coupon.

A 34.0 B 45.4 C 51.1 D 56.7 E 68.1

Question 13.18

An *n*–year bond with semiannual coupons has the following characteristics:

The par value and redemption value are 3,000.

The annual coupon rate is 6% payable semiannually.

The annual yield to maturity is 7% convertible semiannually.

The book value immediately after the 7^{th} coupon is 11 greater than the book value immediately after the 6th coupon.

Calculate *n*.

A 7.5 B 9.4 C 11.3 D 13.1 E 15.0

Question 13.19

A 30-year bond that pays annual coupons is purchased at a discount.

The accumulation of discount in the 12^{th} coupon is 194.49. The accumulation of discount in the 29^{th} coupon is 841.68.

Calculate the amount of discount in the purchase price of the bond.

A 9,076 B 9,475 C 9,875 D 10,274 E 10,673

Question 13.20

An *n*–year bond with annual coupons has the following characteristics:

The redemption value at maturity is 2,000.

The annual effective yield is 5%.

The book value immediately after the 14^{th} coupon is 2,254.60.

The book value immediately after the 15^{th} coupon is 2,223.33.

Calculate *n*.

A 19 B 20 C 21 D 22 E 23

Question 13.21

A 1,000 par value bond with coupons of 8% payable semiannually is called for 1,150 prior to maturity.

The bond was purchased for 988 immediately after a coupon payment. The realized yield on the bond is 9% convertible semiannually.

Calculate the number of years that the bond was held.

A 11 B 14 C 17 D 19 E 22

Question 13.22

A 1,000 par value bond with coupons of 6% payable annually is redeemable at par in 20 years, but it is callable any time from the end of the 10^{th} year at 1,030.

Based on an investor's desired yield rate, the investor calculates the following potential purchase prices:

Assuming the bond is called at the end of the 10^{th} year, the purchase price is 879.69.

Assuming the bond is held until maturity, the purchase price is 803.64.

The investor buys the bond at the highest price that guarantees that the investor will receive at least the desired yield rate.

The investor holds the bond for 15 years, at which time the bond is called.

Calculate the annual effective yield rate that the investor earns.

A 7.0% B 7.5% C 8.0% D 8.5% E 9.0%

Question 13.23

An investor purchases a 12-year callable bond with a face amount of 1,000. The bond has an annual nominal coupon rate of 9% paid semiannually. The price paid for the bond is P.

The bond may be called at par by the issuer on every other coupon payment date, beginning with the second coupon payment date.

The investor earns at least an annual nominal yield of 10% compounded semiannually, regardless of when the bond is redeemed.

Calculate the largest possible value of P.

A 923 B 931 C 939 D 948 E 956

Question 13.24

An investor purchased a 15-year par value bond for 1,733.84. The bond has an annual nominal coupon rate of 7% payable semiannually. The bond can be called at par value X on any coupon date, starting at the end of year 10, after the coupon is paid. The lowest yield rate that the investor can possibly receive is a nominal annual interest rate of 5% convertible semiannually.

Calculate X.

A 1,266 B 1,383 C 1,500 D 1,617 E 1,734

Question 13.25

An investor purchased a 15-year par value bond with semiannual coupons of 50 and a redemption value of 1,200. The bond can be called at 1,300 on any coupon date prior to maturity, starting at the end of year 10.

Calculate the maximum price of the bond that guarantees that the investor will earn an annual nominal interest rate of at least 6% convertible semiannually.

A 1,434 B 1,444 C 1,454 D 1,464 E 1,474

Question 13.26

An investor purchased a 15-year par value bond with semiannual coupons of 50 and a redemption value of 1,200. The bond can be called at 1,400 on any coupon date prior to maturity, starting at the end of year 10.

Calculate the maximum price of the bond that guarantees that the investor will earn an annual nominal interest rate of at least 6% convertible semiannually.

A 1,459 B 1,474 C 1,489 D 1,504 E 1,519

Question 13.27

An investor purchased a 20-year par value bond with an annual coupon rate of 3% payable semiannually. The price of the bond was 1,104.92.

The bond can be called at par value X on any coupon date starting at the end of year 8. The lowest yield rate that the investor can possibly receive is an annual nominal rate of 4% convertible semiannually.

Calculate X.

A 1,270 B 1,275 C 1,280 D 1,285 E 1,290

Question 13.28

An investor purchased a 15-year par value bond for 1,150. The bond has an annual nominal coupon rate of 5% payable semiannually.

The bond can be called at 120 over the par value of 1,200 on any coupon date starting at the end of year 9 and ending six months prior to maturity.

Calculate the minimum yield that the investor could receive, expressed as an annual interest rate that is convertible semiannually.

A 2.7% B 3.4% C 4.1% D 4.7% E 5.4%

Question 13.29

Which of the following statements is/are TRUE?

I. A negative TIPS yield indicates that the cost of inflation protection is greater than the compensation for deferred consumption.

II. Nominal return U.S. Treasury bonds are a risk-free investment for retirees living in the United States.

III. Revenue bonds issued by cities are free of default risk.

A I only B II only C I and II only D II and III only E I and III only

Question 13.30

Which of the following statements is/are TRUE?

I. The issuers of general obligation bonds are obligated to use the revenues from a particular project, such as the tolls received from drivers crossing a bridge, to make the bonds' payments.

II. The interest paid on U.S. municipal bonds is commonly tax-free or taxable at a preferred rate.

III. The yields of the bonds of Canadian provinces and municipalities are lower than the yields of otherwise equivalent bonds issued by the government of Canada.

IV. All bonds issued by the government of Canada are denominated in Canadian dollars.

A I only B II only C I and II only D I, II and III only E II, III, and IV only

Question 13.31

Which of the following statements is TRUE?

A The bid yield of a corporate bond is greater than its ask yield.

B The higher the bid-ask spread of a bond, the more liquid the bond is.

C All else being equal, a bond with a call provision should have a lower yield than a bond without the call provision.

D All else being equal, a bond with a put provision should have a higher yield than a bond without the put provision.

E The yield spread between a corporate bond and a risk-free government bond of the same term and currency is the incremental compensation for credit risk.

Question 13.32

Kelly purchases a U.S. Treasury bill that has a maturity value of 1,000,000, a remaining time until maturity of 95 days, and a quoted rate of 4.5%.

Calculate the dollar amount of interest that Kelly earns on the investment.

A 11,548 B 11,577 C 11,736 D 11,875 E 11,936

Question 13.33

Charlotte purchases a U.S. Treasury bill that has a maturity value of 1,000,000, a remaining time until maturity of 95 days, and a quoted rate of 4.5%.

After 35 days, the quoted rate is 3.6%, and Charlotte sells the Treasury bill.

Assuming no transaction costs and a 365-day year, calculate the annual effective rate of interest that Charlotte earns.

A 3.67% B 4.70% C 6.17% D 6.26% E 6.38%

Question 13.34

Al purchases a Canadian Treasury bill that has a maturity value of 100,000, a remaining time until maturity of 73 days, and a price of 99,304.87.

Let *QR* be the quoted rate on this Canadian Treasury bill.

Let *j* be the annual effective yield on this Canadian T-bill, based on a 365-day year.

Calculate $j - QR$.

A 0.02% B 0.05% C 0.07% D 0.10% E 0.12%

Question 13.35

Ben purchases a Canadian Treasury bill that has a maturity value of 1,000,000, a remaining time until maturity of 95 days, and a quoted rate of 4.5%.

After 35 days, the quoted rate is 3.6%, and Ben sells the Treasury bill.

Assuming no transaction costs and a 365-day year, calculate the annual effective rate of interest that Ben earns.

A 3.67% B 4.70% C 6.17% D 6.26% E 6.38%

Question 13.36

Bill is considering 5 different T-bills that are available for sale. He is not concerned about the exchange rate between U.S. dollars and Canadian dollars because he is able to use a forward agreement to lock in the current exchange rate

All of the T-bills mature for 100,000 in 180 days:

- T-bill A is a U.S. T-bill with a quoted rate of 5%.
- T-bill B is a Canadian T-bill with a quoted rate of 5%.
- T-bill C is a U.S. T-bill with an annual effective yield of 5%.
- T-bill D is a U.S. T-bill with an annual yield of 5% compounded daily.
- T-bill E is a Canadian T-bill with a continuously compounded yield of 5%.

All yields are expressed as annual effective interest rates and based on a 365-day year.

Bill purchases the T-bill with the highest annual effective yield. Which T-bill does Bill purchase?

A T-bill A B T-bill B C T-bill C D T-bill D E T-bill E

Question 13.37

Two Treasury bills each mature for 100,000 in 240 days, and both bills have a current price of 97,000.

One of the Treasury bills is a U.S. Treasury bill, and its maturity value and price are denominated in U.S. dollars.

The other Treasury bill is a Canadian Treasury bill, and its maturity value and price are denominated in Canadian dollars.

Which of the following statements is NOT true?

A The quoted rate for the U.S. T-bill is 4.50%.

B The quoted rate for the Canadian T-bill is greater than the quoted rate for the U.S. T-bill.

C The annual effective interest rate earned on the Canadian T-bill is equal to the annual effective interest rate earned on the U.S. T-bill.

D The annual effective interest rate earned on the U.S. T-bill is greater than the annual quoted rate earned on the U.S. T-bill.

E The annual effective interest rate earned on the Canadian T-bill is less than the quoted rate on the Canadian T-bill.

Chapter 14: Duration and Convexity

When pricing bonds, we found it convenient to set y equal to the yield per unit of time, where the unit of time was equal to the coupon payment period. When evaluating the sensitivity of an asset's price to changes in the yield, however, we are usually interested in changes in the annualized yield. Therefore, in this chapter, we use $y^{(m)}$, where:

$$y^{(m)} = \text{Annual yield, compounded } m \text{ times per year}$$

14.01 Macaulay Duration

Let's use r to denote the continuously compounded annual yield of an asset:

$$r = \lim_{m \to \infty} y^{(m)}$$

The price of an asset is the present value of its cash flows, and it is a function of the continuously compounded yield:

$$P(r) = \sum_{t \geq 0} PV_0(CF_t) = \sum_{t \geq 0} e^{-rt} \times CF_t$$

Suppose that the continuously compounded yield changes by Δr. The Taylor expansion of P about r is:

$$P(r + \Delta r) = P(r) + P'(r) \times \Delta r + \frac{P''(r)}{2}(\Delta r)^2 + \cdots + \frac{P^{(k)}(r)}{k!}(\Delta r)^k + \cdots$$

Using the first two terms on the right side of the equation, we have:

$$P(r + \Delta r) \approx P(r) + P'(r) \times \Delta r$$

$$P(r + \Delta r) - P(r) \approx P'(r) \times \Delta r$$

$$\frac{P(r + \Delta r) - P(r)}{P(r)} \approx \frac{P'(r)}{P(r)} \times \Delta r$$

$$\frac{P(r + \Delta r) - P(r)}{P(r) \times \Delta r} \approx \frac{P'(r)}{P(r)}$$

The portion on the left in the final expression above is the percentage change in the price of the bond per unit change in the yield. The price moves inversely to changes in yield, so the expression is negative. To make it positive, let's multiply by -1. The result is called the **Macaulay duration**:

$$\frac{\%\Delta P}{\Delta r} \approx \frac{P'(r)}{P(r)}$$

$$-\frac{\%\Delta P}{\Delta r} \approx -\frac{P'(r)}{P(r)} = MacD = \text{Macaulay duration}$$

Macaulay duration is named after Frederick Macaulay, an economist who introduced the use of duration for measuring a bond's sensitivity to changes in its yield.

Macaulay duration is sometimes referred to simply as **duration** without the descriptor "Macaulay."

We often measure the change in the yield in **basis points**. A change of 100 basis points moves the yield by 0.01. For example:

- If the yield is 9%, and the yield increases by 100 basis points, then the new yield is 10%.
- If the yield is 9%, and the yield decreases by 20 basis points, then the new yield is 8.8%.

Example 14.01 The Macaulay duration of a bond is 7, and its price is \$102. Estimate the new price of the bond if the continuously compounded yield increases from 5% to 6%.

Solution The change in the continuously compounded yield is 100 basis points:

$$\Delta r = 0.06 - 0.05 = 0.01$$

The percentage change in the price can be estimated by rewriting the formula for Macaulay duration:

$$-\frac{\%\Delta P}{\Delta r} \approx MacD$$

$$\%\Delta P \approx -MacD \times \Delta r$$

$$\%\Delta P \approx -7 \times 0.01$$

$$\%\Delta P \approx -7\%$$

The estimate for the new price is 7% less than the original price:

$$102 \times (1 + \%\Delta P) = 102 \times (1 - 0.07) = \mathbf{94.86}$$

As the example above illustrates, the duration tells us the percentage change in the bond's price resulting from a 100 basis point change. That is, when the duration is 7, we can say that if the yield changes by 100 basis points, then the price of the bond will change by 7%.

Recall that the price of the asset is a function of r:

$$P(r) = \sum_{t \geq 0} e^{-rt} \times CF_t$$

$$P'(r) = \frac{dP(r)}{dr} = -\sum_{t \geq 0} te^{-rt} \times CF_t$$

The Macaulay duration is:

$$MacD = -\frac{P'(r)}{P(r)} = \frac{\sum_{t \geq 0} te^{-rt} \times CF_t}{P(r)} = \frac{\sum_{t \geq 0} te^{-rt} \times CF_t}{\sum_{t \geq 0} e^{-rt} \times CF_t} = \frac{\sum_{t \geq 0} \left[t \times PV_0(CF_t) \right]}{\sum_{t \geq 0} PV_0(CF_t)}$$

The rightmost expression above tells us that the Macaulay duration is the weighted average of the times that the cash flows occur, where the weights are the present values of the cash flows.

14.01

Macaulay Duration

The Macaulay duration of a financial instrument is the weighted average of the timing of the cash flows:

$$MacD = -\frac{P'(r)}{P(r)} = \frac{\sum_{t \geq 0} \left[t \times PV_0(CF_t) \right]}{\sum_{t \geq 0} PV_0(CF_t)}$$

The Macaulay duration can be used to estimate the percentage change in price resulting from a shift in the continuously compounded yield:

$$\%\Delta P \approx -MacD \times \Delta r$$

We don't have to find r to determine the MacD. We can use a yield that is not compounded continuously to find the present value of the cash flows.

Example 14.02 A 2-year bond pays semiannual coupons at an annual rate of 6% per year. The face amount of the bond is \$100. The yield of the bond is 8% compounded semiannually.

Calculate the Macaulay duration of the bond.

Solution The price of the bond is:

$$\sum_{t\geq 0} PV_0\left(CF_t\right) = \frac{3}{1.04} + \frac{3}{1.04^2} + \frac{3}{1.04^3} + \frac{103}{1.04^4} = 96.3701$$

The numerator of the formula for the Macaulay duration is:

$$\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right] = \frac{3\times 0.5}{1.04} + \frac{3\times 1}{1.04^2} + \frac{3\times 1.5}{1.04^3} + \frac{103\times 2}{1.04^4} = 184.3061$$

Alternatively, we can use an increasing annuity-immediate to find the value of the numerator:

$$\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right] = 3\times 0.5\left[\frac{1}{1.04} + \frac{2}{1.04^2} + \frac{3}{1.04^3} + \frac{4}{1.04^4}\right] + \frac{100\times 2}{1.04^4}$$

$$= 3\times 0.5\times (Ia)_{\overline{4}|0.04} + \frac{100\times 2}{1.04^4} = 1.5\times \frac{3.7751 - \frac{4}{1.04^4}}{0.04} + 170.9608$$

$$= 1.5\times 8.8969 + 170.9608 = 184.3061$$

The Macaulay duration is:

$$MacD = \frac{\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t\geq 0} PV_0\left(CF_t\right)} = \frac{184.3061}{96.3701} = \mathbf{1.9125}$$

If the timing an asset's cash flows is pushed back by k years, then the asset's Macaulay duration is increased by k years. This is can be shown by observing that the weights do not change. Let's begin with the original timing of the cash flows:

$$MacD_{Original} = \frac{\sum_{t\geq 0} te^{-rt}\times CF_t}{\sum_{t\geq 0} e^{-rt}\times CF_t}$$

Now suppose that each cash flow is moved, so that it occurs k years later. The new Macaulay duration is:

$$MacD = \frac{\sum_{t\geq 0}(t+k)e^{-r(t+k)}\times CF_t}{\sum_{t\geq 0} e^{-r(t+k)}\times CF_t} = \frac{e^{-rk}\sum_{t\geq 0}(t+k)e^{-rt}\times CF_t}{e^{-rk}\sum_{t\geq 0} e^{-rt}\times CF_t}$$

$$= \frac{\sum_{t\geq 0}(t+k)e^{-rt}\times CF_t}{\sum_{t\geq 0} e^{-rt}\times CF_t}$$

As shown in the final expression above, the weights have not changed, but the timing of each cash flow now has k added to it:

$$MacD = \frac{\sum_{t\geq 0}(t+k)e^{-rt}\times CF_t}{\sum_{t\geq 0} e^{-rt}\times CF_t} = MacDur_{Original} + k$$

If the cash flows are instead adjusted to occur earlier, then the value of k in the expression above becomes negative.

The example below uses the same bond as the previous example, but it pushes each of the cash flows back by 3 years. Therefore, the Macaulay duration can be found by adding 3 to the previous solution:

$$MacD = MacD_{Original} + k = 1.9125 + 3 = 4.9125$$

Below, we use the basic formula to obtain the same Macaulay duration.

Example 14.03 A deferred 2-year bond pays semiannual coupons at an annual rate of 6% per year. The bond payments are deferred for 3 years, so the first payment occurs at the end of 3.5 years, and the final payment occurs at the end of 5 years. The face amount of the bond is $100. The yield of the bond is 8% compounded semiannually.

Calculate the Macaulay duration of the bond.

Solution The price of the bond is:

$$\sum_{t\geq 0} PV_0\left(CF_t\right) = \frac{3}{1.04^7} + \frac{3}{1.04^8} + \frac{3}{1.04^9} + \frac{103}{1.04^{10}} = 76.1627$$

The numerator of the formula for the Macaulay duration is:

$$\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right] = \frac{3\times 3.5}{1.04^7} + \frac{3\times 4}{1.04^8} + \frac{3\times 4.5}{1.04^9} + \frac{103\times 5}{1.04^{10}} = 374.1479$$

The Macaulay duration is:

$$MacD = \frac{\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t\geq 0} PV_0\left(CF_t\right)} = \frac{374.1479}{76.1627} = \mathbf{4.9125}$$

A **zero-coupon bond** does not pay any coupons, and it only makes one payment, which occurs when the bond matures. Since a zero-coupon bond has only one cash flow, the Macaulay duration of a zero-coupon bond is the maturity of the bond. If a zero-coupon bond matures at time n, then its Macaulay duration is:

$$MacD = \frac{\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t\geq 0} PV_0\left(CF_t\right)} = \frac{n \times PV_0(CF_n)}{PV_0(CF_n)} = n$$

14.02

Macaulay Duration of a Zero-Coupon Bond

If a zero-coupon bond matures at time n, then its Macaulay duration is n.

Example 14.04 A 10-year bond with a par value of $100 is a zero-coupon bond. The yield of the bond is 8% compounded semiannually.

Calculate the Macaulay duration of the bond.

Solution The Macaulay duration is:

$$MacD = \frac{\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t\geq 0} PV_0\left(CF_t\right)} = \frac{10 \times PV_0\left(CF_{10}\right)}{PV_0\left(CF_{10}\right)} = \mathbf{10}$$

In the solution to Example 14.02, we saw that we can use the formula for the present value of an increasing annuity-immediate to find the duration of a bond. Consider a bond that pays coupons m times per year for n years and matures for its face value. The Macaulay duration of the bond is:

$$MacD = \frac{\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t\geq 0} PV_0\left(CF_t\right)} = \frac{Coup\left[\frac{1}{m}v + \frac{2}{m}v^2 + \cdots + \frac{mn}{m}v^{mn}\right] + nFv^{mn}}{Coup\left[v + v^2 + \cdots + v^{mn}\right] + Fv^{mn}}$$

$$= \frac{\frac{Coup}{m} \times (Ia)_{\overline{mn}|y} + nFv^{mn}}{Coup \times a_{\overline{mn}|y} + Fv^{mn}}$$

where:

$$y = \frac{y^{(m)}}{m} \qquad \text{and} \qquad v = \frac{1}{1+y}$$

If the bond is a par bond, then its price is F, its coupon rate is equal to its yield, and its periodic coupon is equal to its face value times its periodic effective yield:

$$c = y \qquad \Rightarrow \qquad Coup = F \times y$$

The Macaulay duration of a par bond that matures at time n is:

$$MacD = \frac{\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t\geq 0} PV_0\left(CF_t\right)} = \frac{\frac{Coup}{m} \times (Ia)_{\overline{mn}|y} + nFv^{mn}}{Coup \times a_{\overline{mn}|y} + Fv^{mn}}$$

$$= \frac{\frac{F \times y}{m} \times (Ia)_{\overline{mn}|y} + nFv^{mn}}{F} = \frac{y}{m} \times (Ia)_{\overline{mn}|y} + nv^{mn}$$

$$= \frac{y}{m} \times \frac{\ddot{a}_{\overline{nm}|y} - nmv^{mn}}{y} + nv^{mn} = \frac{1}{m} \times \left[\ddot{a}_{\overline{nm}|y} - nmv^{mn}\right] + nv^{mn}$$

$$= \frac{1}{m} \times \ddot{a}_{\overline{nm}|y} = \ddot{a}^{(m)}_{\overline{n}|}$$

Macaulay Duration of a Par Bond

14.03

A bond that is priced at a par, pays coupons m times per year, and matures at time n has a Macaulay duration that is equal to the present value of an annuity-due:

$$MacD = \ddot{a}^{(m)}_{\overline{n}|}$$

The interest rate used to find the present value of the annuity-due is the bond's yield.

Example 14.05

A 15-year bond has a par value of \$100, and it makes coupon payments of \$4 every 6 months. The yield of the bond is 8% compounded semiannually.

Calculate the Macaulay duration of the bond.

Solution

Since the annual coupon rate is equal to the yield of 8%, the bond is a par bond. The Macaulay duration is:

$$MacD = \ddot{a}^{(2)}_{\overline{15}|} = \frac{1}{2} \times \frac{1 - 1.04^{-30}}{\frac{0.04}{1.04}} = \mathbf{8.9919}$$

Suppose that a portfolio consists of k bonds, with prices $P_j(r)$ for $j \in \{1,2,\cdots,k\}$. The value of the portfolio is:

$$Port = \sum_{j=1}^{k} P_j(r)$$

Contrary to our usual convention, the subscript j indicates which bond we are considering, not the time at which the bond is being valued.

If all of the bonds in the portfolio have the same yield, then the Macaulay duration of the portfolio is:

$$\begin{aligned}
MacD_{Port} &= -\frac{d[Port]}{dr} \times \frac{1}{Port} = -\frac{1}{Port}\sum_{j=1}^{k} P_j{}'(r) \\
&= -\frac{P_1(r)}{Port} \times \frac{P_1{}'(r)}{P_1(r)} - \frac{P_2(r)}{Port} \times \frac{P_1{}'(r)}{P_2(r)} - \cdots - \frac{P_k(r)}{Port} \times \frac{P_1{}'(r)}{P_k(r)} \\
&= w_1 \times MacD_1 + w_2 \times MacD_2 + \cdots + w_k \times MacD_k \\
&= \sum_{j=1}^{k} w_j \times MacD_j
\end{aligned}$$

where:

$$MacD_j = \text{Macaulay duration of the } j^{\text{th}} \text{ asset}$$

$$w_j = \frac{\text{Present value of } j^{\text{th}} \text{ asset}}{\text{Present value of the portfolio}}$$

The duration of the portfolio is the weighted average of the durations of its component assets, with the weights being the value of each component asset.

14.04

Macaulay Duration of a Portfolio

The Macaulay duration of a portfolio consisting of k assets is the weighted average of the Macaulay durations of the assets:

$$MacD_{Port} = \sum_{j=1}^{k} w_j \times MacD_j$$

where:

$$MacD_j = \text{Macaulay duration of the } j^{\text{th}} \text{ asset}$$

$$w_j = \frac{\text{Present value of } j^{\text{th}} \text{ asset}}{\text{Present value of the portfolio}}$$

Example 14.06

A portfolio consists of two bonds. The first bond is a zero-coupon bond that matures for \$1,000 in 23 years. The second bond is a 15-year bond with a par value of \$100, and it makes coupon payments of \$4 every 6 months. The yield of both bonds is 8% compounded semiannually.

Calculate the Macaulay duration of the portfolio.

Solution

The duration of the zero-coupon bond is 23.

The 15-year bond is a par bond, so its Macaulay duration is:

$$MacD = \ddot{a}_{\overline{15}|}^{(2)} = \frac{1}{2} \times \frac{1 - 1.04^{-30}}{\frac{0.04}{1.04}} = 8.9919$$

The duration of the portfolio is:

$$MacD_{Port} = \sum_{j=1}^{k} w_j \times MacD_j = \frac{1,000 \times \dfrac{1}{1.04^{46}} \times 23 + 100 \times 8.9919}{1,000 \times \dfrac{1}{1.04^{46}} + 100}$$

$$= \mathbf{17.7062}$$

14.02 Modified Duration

The price of a bond can be graphed as a function of the bond's yield:

Suppose that the annual yield of an asset is compounded m times per year and is $y^{(m)}$. The price of the asset is the present value of its cash flows, which is a function of $y^{(m)}$:

$$P\left(y^{(m)}\right) = \sum_{t \geq 0} PV_0(CF_t) = \sum_{t \geq 0} CF_t \left(1 + \frac{y^{(m)}}{m}\right)^{-tm}$$

The derivative of the price is:

$$P'\left(y^{(m)}\right) = \frac{dP\left(y^{(m)}\right)}{dy^{(m)}} = \sum_{t \geq 0} \left[-t \times CF_t \times \left(1 + \frac{y^{(m)}}{m}\right)^{-tm-1}\right]$$

$$= -\left(1 + \frac{y^{(m)}}{m}\right)^{-1} \sum_{t \geq 0} \left[t \times CF_t \times \left(1 + \frac{y^{(m)}}{m}\right)^{-tm}\right]$$

$$= -\frac{\sum_{t \geq 0} \left[t \times PV_0(CF_t)\right]}{\left(1 + \dfrac{y^{(m)}}{m}\right)}$$

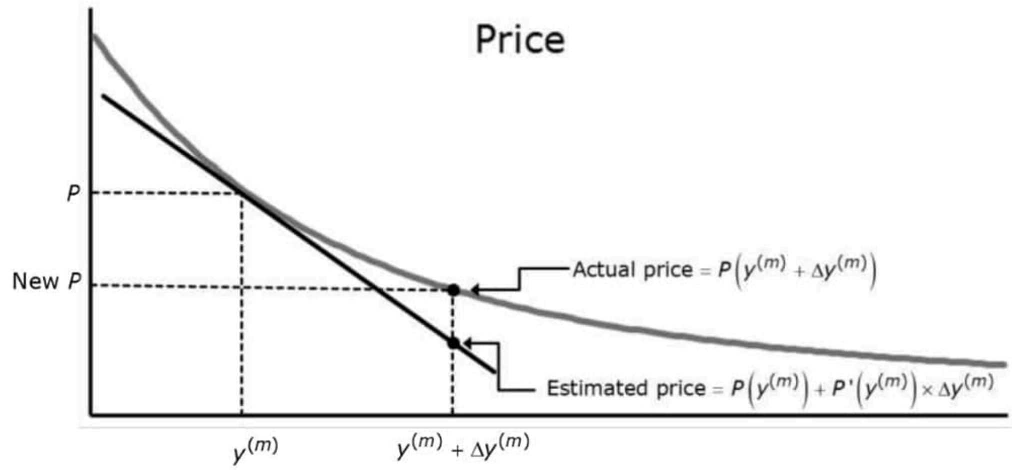
Suppose that the yield changes by $\Delta y^{(m)}$. The Taylor expansion of P about $y^{(m)}$ is:

$$P\left(y^{(m)} + \Delta y^{(m)}\right) = P\left(y^{(m)}\right) + P'\left(y^{(m)}\right) \times \Delta y^{(m)} + \frac{P''\left(y^{(m)}\right)}{2}\left(\Delta y^{(m)}\right)^2 + \cdots$$
$$+ \frac{P^{(k)}\left(y^{(m)}\right)}{k!}\left(\Delta y^{(m)}\right)^k + \cdots$$

Using the first two terms on the right side of the equation, we have an estimate for the new price resulting from the change in the yield:

$$P\left(y^{(m)}+\Delta y^{(m)}\right)\approx P\left(y^{(m)}\right)+P'\left(y^{(m)}\right)\times\Delta y^{(m)}$$

As shown below, the estimated price is less than the actual price because the price curve is convex.



Rearranging the expression for the estimated price, we have:

$$P\left(y^{(m)}+\Delta y^{(m)}\right)\approx P\left(y^{(m)}\right)+P'\left(y^{(m)}\right)\times\Delta y^{(m)}$$

$$P\left(y^{(m)}+\Delta y^{(m)}\right)-P\left(y^{(m)}\right)\approx P'\left(y^{(m)}\right)\times\Delta y^{(m)}$$

$$\frac{P\left(y^{(m)}+\Delta y^{(m)}\right)-P\left(y^{(m)}\right)}{P\left(y^{(m)}\right)\times\Delta y^{(m)}}\approx\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)}$$

The portion on the left in the final expression above is the percentage change in the price of the bond per unit change in the yield. The price moves inversely to changes in yield, so the expression is negative. To make it positive, let's multiply by -1. The result is called **modified duration**:

$$\frac{\%\Delta P}{\Delta y^{(m)}}\approx\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)}$$

$$-\frac{\%\Delta P}{\Delta y^{(m)}}\approx-\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)}=ModD=\text{Modified duration}$$

Example 14.07 The yield of a bond is expressed as a rate that is compounded twice per year, and the bond's modified duration is 8. The price of the bond is \$98. Estimate the new price of the bond if the yield decreases from 7% to 5.5%.

Solution The change in the yield is -150 basis points:

$$\Delta y^{(2)}=0.055-0.070=-0.015$$

The percentage change in the price can be estimated by rewriting the formula for modified duration:

$$-\frac{\%\Delta P}{\Delta y^{(m)}} \approx ModD$$

$$\%\Delta P \approx -ModD \times \Delta y^{(m)}$$

$$\%\Delta P \approx -8 \times (-0.015)$$

$$\%\Delta P \approx 12\%$$

The estimate for the new price is 12% greater than the original price:

$$98 \times (1 + \%\Delta P) = 98 \times (1 + 0.12) = \mathbf{109.76}$$

The modified duration is a modification of the Macaulay duration because it is the Macaulay duration divided by an accrual factor for a period of length $1/m$:

$$ModD = -\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} = \frac{\sum_{t\geq 0}\left[t \times PV_0(CF_t)\right]}{\left(1+\frac{y^{(m)}}{m}\right)P\left(y^{(m)}\right)} = \frac{MacD}{1+\frac{y^{(m)}}{m}}$$

14.05

Modified Duration

The modified duration of a financial instrument is the Macaulay duration divided by a one-period accrual factor:

$$ModD = -\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} = \frac{\sum_{t\geq 0}\left[t \times PV_0(CF_t)\right]}{\left(1+\frac{y^{(m)}}{m}\right)P\left(y^{(m)}\right)} = \frac{MacD}{1+\frac{y^{(m)}}{m}}$$

The modified duration can be used to estimate the percentage change in price resulting from a shift in the yield $y^{(m)}$:

$$\%\Delta P \approx -ModD \times \Delta y^{(m)}$$

The price found using the modified duration to estimate the $\%\Delta P$ is known as the **first-order modified approximation.** For convenience, we refer to the first-order modified approximation as the FoMod approximation:

$$P\left(y^{(m)} + \Delta y^{(m)}\right) \approx P\left(y^{(m)}\right) \times \left[1 - ModD \times \Delta y^{(m)}\right]$$

In Example 14.07, we found the first-order modified approximation to be 109.76.

We can obtain the modified duration by first finding the Macaulay duration.

Example 14.08

A 2-year bond pays semiannual coupons at an annual rate of 6% per year. The face amount of the bond is $100. The yield of the bond is 8% compounded semiannually.

Calculate the modified duration of the bond.

Solution

The price of the bond is:

$$\sum_{t\geq 0} PV_0\left(CF_t\right) = \frac{3}{1.04} + \frac{3}{1.04^2} + \frac{3}{1.04^3} + \frac{103}{1.04^4} = 96.3701$$

The numerator of the formula for the Macaulay duration is:

$$\sum_{t\geq 0}\left[t \times PV_0\left(CF_t\right)\right] = \frac{3 \times 0.5}{1.04} + \frac{3 \times 1}{1.04^2} + \frac{3 \times 1.5}{1.04^3} + \frac{103 \times 2}{1.04^4} = 184.3061$$

The Macaulay duration is:

$$MacD = \frac{\sum_{t \geq 0} \left[t \times PV_0 \left(CF_t \right) \right]}{\sum_{t \geq 0} PV_0 \left(CF_t \right)} = \frac{184.3061}{96.3701} = 1.9125$$

The modified duration is:

$$ModD = \frac{MacD}{1 + \frac{y^{(m)}}{m}} = \frac{1.9125}{1 + \frac{0.08}{2}} = \frac{1.9125}{1.04} = \mathbf{1.8389}$$

The bond worksheet in the BA II Plus Professional can be used to calculate modified duration:

[2nd] [BOND] 12.3100 [ENTER] ↓
6 [ENTER] ↓ (enter the coupon rate)
12.3102 [ENTER] ↓
100 [ENTER] ↓
(Use [2nd] [SET] until display shows 360) ↓
(Use [2nd] [SET] until display shows 2/Y) ↓
8 [ENTER] ↓
[CPT] ↓ ↓
Result is **1.8389**.

Unfortunately, the duration feature is not available in the non-professional edition of the BA II Plus.

There seems to be a bug in the duration calculation in the BA II Plus Professional. When the redemption value is set to a value other than 100, the duration calculation calculates the duration as if the redemption value is 100. This is only a problem when the redemption value is not equal to the face value.

Although it is common to find the modified duration of a bond by first finding the Macaulay duration and then dividing by a one-period accrual factor, there are some assets for which it is easier to calculate the modified duration directly.

Example 14.09 A perpetuity pays \$100 at the end of each year. The price of the perpetuity is \$1,000. Find the modified duration of the perpetuity with respect to the annual effective yield.

Solution The price of the perpetuity is the amount of each payment divided by its yield:

$$1{,}000 = \frac{100}{y}$$

$$y = 0.10$$

The first derivative of the price is found below:

$$P(y) = \frac{100}{y}$$

$$P'(y) = -100y^{-2}$$

The modified duration is:

$$ModD = -\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} = -\frac{-100y^{-2}}{100y^{-1}} = \frac{1}{y} = \frac{1}{0.10} = \mathbf{10}$$

The modified duration of a par bond that pays coupons m times per year and matures at time n is equal to the present value of an annuity-immediate:

$$ModD = \frac{MacD}{1+\frac{y^{(m)}}{m}} = \frac{\ddot{a}_{\overline{n}|}^{(m)}}{1+\frac{y^{(m)}}{m}} = a_{\overline{n}|}^{(m)}$$

Modified Duration of a Par Bond

14.06 A bond that is priced at a par, pays coupons m times per year, and matures at time n has a modified duration that is equal to the present value of an annuity-immediate:

$$ModD = a_{\overline{n}|}^{(m)}$$

The interest rate used to find the present value of the annuity-immediate is the bond's yield.

Example 14.10 A 15-year bond has a par value of \$100, and it makes coupon payments of \$4 every 6 months. The yield of the bond is 8% compounded semiannually.

Calculate the modified duration of the bond.

Solution Since the annual coupon rate is equal to the yield of 8%, the bond is a par bond. The modified duration is:

$$ModD = a_{\overline{15}|}^{(2)} = \frac{1}{2} \times \frac{1-1.04^{-30}}{0.04} = \mathbf{8.6460}$$

The Macaulay duration is used to estimate the percentage price change from a shift in the continuously compounded yield, while the modified duration is used to estimate the percentage price change from a shift in a yield that is compounded m times per year. As m becomes larger, the modified duration approaches the Macaulay duration:

$$\lim_{m\to\infty} ModD = \lim_{m\to\infty} \frac{MacD}{1+\frac{y^{(m)}}{m}} = MacD$$

Macaulay duration is just a special case of modified duration, where the compounding frequency of the yield is infinite.

Since the Macaulay duration of a portfolio is the weighted average of the Macaulay durations of its components, the same is true for modified duration.

$$ModD_{Port} = \frac{MacD_{Port}}{1+\frac{y^{(m)}}{m}} = \sum_{j=1}^{k} w_j \times \frac{MacD_j}{1+\frac{y^{(m)}}{m}} = \sum_{j=1}^{k} w_j \times ModD_j$$

Modified Duration of a Portfolio

14.07 The modified duration of a portfolio consisting of k assets is the weighted average of the modified durations of the assets:

$$ModD_{Port} = \sum_{j=1}^{k} w_j \times ModD_j$$

where:

$$ModD_j = \text{Modified duration of the } j^{\text{th}} \text{ asset}$$

$$w_j = \frac{\text{Present value of } j^{\text{th}} \text{ asset}}{\text{Present value of the portfolio}}$$

Example 14.11 A portfolio consists of two bonds. The first bond is a zero-coupon bond that matures for \$1,000 in 23 years. The second bond is a 15-year bond with a par value of \$100, and it makes coupon payments of \$4 every 6 months. The yield of both bonds is 8% compounded semiannually.

Calculate the modified duration of the portfolio.

Solution The Macaulay duration of the zero-coupon bond is 23.

The 15-year bond is a par bond, so its Macaulay duration is:

$$MacD = \ddot{a}_{\overline{15}|}^{(2)} = \frac{1}{2} \times \frac{1 - 1.04^{-30}}{\frac{0.04}{1.04}} = 8.9919$$

There are two ways to find the modified duration.

First, we find the Macaulay duration and then adjust it to obtain the modified duration. The Macaulay duration of the portfolio is:

$$MacD_{Port} = \sum_{j=1}^{k} w_j \times MacD_j = \frac{1{,}000 \times \frac{1}{1.04^{46}} \times 23 + 100 \times 8.9919}{1{,}000 \times \frac{1}{1.04^{46}} + 100}$$

$$= 17.7062$$

The modified duration of the portfolio is:

$$ModD_{Port} = \frac{MacD_{Port}}{1 + \frac{y^{(m)}}{m}} = \frac{17.7062}{1 + \frac{0.08}{2}} = \frac{17.7062}{1.04} = \mathbf{17.0252}$$

Alternatively, we find the modified durations of the components of the portfolio and then find their weighted average. The modified durations of the 23-year bond and the 15-year bond are:

$$ModD_{23} = \frac{23}{1.04} = 22.1154$$

$$ModD_{15} = \frac{8.9919}{1.04} = 8.6460$$

The weighted average of the modified durations is:

$$ModD_{Port} = \sum_{j=1}^{k} w_j \times ModD_j = \frac{1{,}000 \times \frac{1}{1.04^{46}} \times 22.1154 + 100 \times 8.6460}{1{,}000 \times \frac{1}{1.04^{46}} + 100}$$

$$= \mathbf{17.0252}$$

14.03 Convexity

The Taylor expansion of P about $y^{(m)}$ is:

$$P\left(y^{(m)} + \Delta y^{(m)}\right) = P\left(y^{(m)}\right) + P'\left(y^{(m)}\right) \times \Delta y^{(m)} + \frac{P''\left(y^{(m)}\right)}{2}\left(\Delta y^{(m)}\right)^2 + \cdots$$
$$+ \frac{P^{(k)}\left(y^{(m)}\right)}{k!}\left(\Delta y^{(m)}\right)^k + \cdots$$

Using the first two terms to the right of the equal sign, we obtained the following approximation based on modified duration:

$$\%\Delta P \approx -ModD \times \Delta y^{(m)} \qquad \text{where: } ModD = -\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)}$$

Let's use an additional term to obtain a better estimate for the percentage change in the price. In the derivation below, *ModC* stands for **modified convexity**:

$$P\left(y^{(m)} + \Delta y^{(m)}\right) \approx P\left(y^{(m)}\right) + P'\left(y^{(m)}\right) \times \Delta y^{(m)} + \frac{P''\left(y^{(m)}\right)}{2}\left(\Delta y^{(m)}\right)^2$$

$$P\left(y^{(m)} + \Delta y^{(m)}\right) - P\left(y^{(m)}\right) \approx P'\left(y^{(m)}\right) \times \Delta y^{(m)} + \frac{P''\left(y^{(m)}\right)}{2}\left(\Delta y^{(m)}\right)^2$$

$$\frac{P\left(y^{(m)} + \Delta y^{(m)}\right) - P\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} \approx \frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} \times \Delta y^{(m)} + \frac{1}{2} \times \frac{P''\left(y^{(m)}\right)}{P\left(y^{(m)}\right)}\left(\Delta y^{(m)}\right)^2$$

$$\%\Delta P \approx -ModD \times \Delta y^{(m)} + 0.5 \times ModC \times \left(\Delta y^{(m)}\right)^2$$

where:

$$ModC = \frac{P''\left(y^{(m)}\right)}{P\left(y^{(m)}\right)}$$

The modified convexity of an asset is the second derivative of the asset's price with respect to the yield, divided by the asset's price. Modified convexity is usually described as just the convexity without the descriptor "modified."

Duration refers to Macaulay duration, but convexity refers to modified convexity. It's not consistent, but that's the way it is!

Example 14.12 The yield of a bond is expressed as a rate that is compounded twice per year, and the bond's modified duration is 8. The modified convexity of the bond is 75. The price of the bond is $98. Use the modified duration and modified convexity to estimate the new price of the bond if the yield decreases from 7% to 5.5%.

Solution The change in the yield is −150 basis points:

$$\Delta y^{(2)} = 0.055 - 0.070 = -0.015$$

The percentage change in the price can be estimated by rewriting the formula for modified duration:

$$\%\Delta P \approx -ModD \times \Delta y^{(m)} + 0.5 \times ModC \times \left(\Delta y^{(m)}\right)^2$$

$$\%\Delta P \approx -8 \times (-0.015) + 0.5 \times 75 \times (-0.015)^2$$

$$\%\Delta P \approx 12.84\%$$

The estimate for the new price is 12.84% greater than the original price:

$$98 \times (1 + \%\Delta P) = 98 \times (1 + 0.1284) = \mathbf{110.59}$$

Since the convexity term uses the square of the change in the yield, including convexity when estimating a new price increases the estimate, regardless of whether the yield has increased or decreased.

Modified Convexity

14.08 The modified convexity of an asset is the second derivative of the asset's price with respect to the asset's yield, divided by the asset's price:

$$ModC = \frac{P''\left(y^{(m)}\right)}{P\left(y^{(m)}\right)}$$

The modified convexity can be used to estimate the percentage change in price resulting from a shift in the yield $y^{(m)}$:

$$\%\Delta P \approx -ModD \times \Delta y^{(m)} + 0.5 \times ModC \times \left(\Delta y^{(m)}\right)^2$$

In the two examples below, we obtain a formula for the price of the asset, and then we take the first and second derivative of the formula with respect to the asset's yield.

Example 14.13 A perpetuity pays \$100 at the end of each year. The price of the perpetuity is \$1,000. Find the modified convexity of the bond with respect to the annual effective yield.

Solution The price of the perpetuity is the amount of each payment divided by its yield:

$$1{,}000 = \frac{100}{y}$$

$$y = 0.10$$

The second derivative of the price is found below:

$$P(y) = \frac{100}{y}$$

$$P'(y) = -100y^{-2}$$

$$P''(y) = 200y^{-3}$$

The modified convexity is:

$$ModC = \frac{P''\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} = \frac{200y^{-3}}{100y^{-1}} = 2y^{-2} = \frac{2}{0.10^2} = \mathbf{200}$$

In the next example, we find the modified convexity of a bond.

Example 14.14 A 3-year bond pays semiannual coupons at an annual rate of 5% per year. The face amount of the bond is \$100. The annual effective yield of the bond is 7%.

Calculate the modified convexity of the bond.

Solution The price of the bond is:

$$P(y) = \frac{5}{1+y} + \frac{5}{(1+y)^2} + \frac{105}{(1+y)^3}$$

At the yield of 7%, the price is:

$$P(0.07) = \frac{5}{1.07} + \frac{5}{(1.07)^2} + \frac{105}{(1.07)^3} = 94.7514$$

The first derivative of the price is:

$$P'(y) = \frac{-5}{(1+y)^2} + \frac{-10}{(1+y)^3} + \frac{-315}{(1+y)^4}$$

The second derivative of the price is:

$$P''(y) = \frac{10}{(1+y)^3} + \frac{30}{(1+y)^4} + \frac{1,260}{(1+y)^5}$$

At the yield of 7%, the second derivative of the price is:

$$P''(y) = \frac{10}{(1.07)^3} + \frac{30}{(1.07)^4} + \frac{1,260}{(1.07)^5} = 929.4124$$

The modified convexity is:

$$ModC = \frac{P''(y)}{P(y)} = \frac{929.4124}{94.7514} = \mathbf{9.8090}$$

Macaulay convexity (*MacC*) is the second derivative of the asset's price with respect to the asset's continuously compounded yield:

$$MacC = \frac{P''(r)}{P(r)}$$

Macaulay convexity is just a special case of modified convexity, where the compounding frequency of the yield is infinite.

The Macaulay convexity can be written as the weighted average square of the timing of each cash flow:

$$MacC = \frac{P''(r)}{P(r)} = \frac{\sum_{t \geq 0} t^2 e^{-rt} \times CF_t}{\sum_{t \geq 0} e^{-rt} \times CF_t} = \frac{\sum_{t \geq 0} \left[t^2 \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)}$$

The weighted average square of the difference between the timing of the cash flows and the Macaulay duration is known as the **dispersion**:

$$Dispersion = \frac{\sum_{t \geq 0} \left[(MacD - t)^2 \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)}$$

The dispersion can be rewritten as:

$$\begin{aligned}
Dispersion &= \frac{\sum_{t \geq 0} \left[(MacD - t)^2 \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)} \\
&= \frac{\sum_{t \geq 0} \left[\left(MacD^2 - 2t \times MacD + t^2\right) \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)} \\
&= MacD^2 - \frac{2 \times MacD \sum_{t \geq 0} \left[t \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)} + \frac{\sum_{t \geq 0} \left[t^2 \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)} \\
&= MacD^2 - 2 \times MacD \times MacD + MacC \\
&= MacC - MacD^2
\end{aligned}$$

14.09

Macaulay Convexity

The Macaulay convexity of an asset is the weighted average of the squares of the times that the cash flows occur:

$$MacC = \frac{P''(r)}{P(r)} = \frac{\sum_{t \geq 0}\left[t^2 \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)}$$

The dispersion of an asset's cash flows is the weighted average of the squares of the differences between the times that the cash flows occur and the Macaulay duration:

$$Dispersion = \frac{\sum_{t \geq 0}\left[(MacD - t)^2 \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)}$$

The Macaulay convexity is equal to the square of the Macaulay duration plus the dispersion:

$$MacC = MacD^2 + Dispersion$$

The more dispersed an asset's cash flows are, the greater the asset's convexity.

Example 14.15

A 2-year bond pays semiannual coupons at an annual rate of 6% per year. The face amount of the bond is $100. The yield of the bond is 8% compounded semiannually.

a. Calculate the Macaulay convexity of the bond.
b. Calculate the dispersion of the bond.

Solution

a. The price of the bond is:

$$\sum_{t \geq 0} PV_0(CF_t) = \frac{3}{1.04} + \frac{3}{1.04^2} + \frac{3}{1.04^3} + \frac{103}{1.04^4} = 96.3701$$

The numerator of the formula for Macaulay convexity is:

$$\sum_{t \geq 0}\left[t^2 \times PV_0(CF_t)\right] = \frac{3 \times 0.5^2}{1.04} + \frac{3 \times 1^2}{1.04^2} + \frac{3 \times 1.5^2}{1.04^3} + \frac{103 \times 2^2}{1.04^4} = 361.6748$$

The Macaulay convexity is:

$$MacC = \frac{P''(r)}{P(r)} = \frac{\sum_{t \geq 0}\left[t^2 \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)} = \frac{361.6748}{96.3701} = \mathbf{3.7530}$$

b. The numerator of the formula for Macaulay duration is:

$$\sum_{t \geq 0}\left[t \times PV_0(CF_t)\right] = \frac{3 \times 0.5}{1.04} + \frac{3 \times 1}{1.04^2} + \frac{3 \times 1.5}{1.04^3} + \frac{103 \times 2}{1.04^4} = 184.3061$$

The Macaulay duration is:

$$MacD = \frac{\sum_{t \geq 0}\left[t \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)} = \frac{184.3061}{96.3701} = 1.9125$$

The dispersion of the bond is found below:

$$MacC = MacD^2 + Dispersion$$

$$3.7530 = 1.9125^2 + Dispersion$$

$$Dispersion = \mathbf{0.09539}$$

The Macaulay convexity of a zero-coupon bond is equal to the square of the bond's Macaulay duration.

Example 14.16

A zero-coupon bond matures for $1,000 in 23 years. The yield of the bond is 8% compounded semiannually.

Calculate the Macaulay convexity of the bond.

Solution

The Macaulay duration of the zero-coupon bond is 23.

The dispersion of the zero-coupon bond is zero:

$$Dispersion = \frac{\sum_{t\geq 0}\left[(MacD - t)^2 \times PV_0\left(CF_t\right)\right]}{\sum_{t\geq 0} PV_0\left(CF_t\right)} = \frac{(23-23)^2 \times PV_0\left(CF_{23}\right)}{PV_0\left(CF_{23}\right)} = 0$$

The Macaulay convexity of a zero-coupon bond is the square of its Macaulay duration:

$$MacC = MacD^2 + Dispersion = 23^2 + 0 = \mathbf{529}$$

We can express the modified convexity in terms of the Macaulay duration and Macaulay convexity. Let's begin by expressing the price of an asset as the present value of its future cash flows:

$$P\left(y^{(m)}\right) = \sum_{t\geq 0} PV_0(CF_t) = \sum_{t\geq 0} CF_t\left(1 + \frac{y^{(m)}}{m}\right)^{-tm}$$

The first derivative of the price of the asset is:

$$P'\left(y^{(m)}\right) = \sum_{t\geq 0}\left[-tm \times CF_t \times \left(1 + \frac{y^{(m)}}{m}\right)^{-tm-1} \times \frac{1}{m}\right] = -\frac{\sum_{t\geq 0}\left[t \times PV_0(CF_t)\right]}{\left(1 + \frac{y^{(m)}}{m}\right)}$$

The second derivative of the price of the asset is:

$$P''\left(y^{(m)}\right) = \sum_{t\geq 0}\left[t(tm+1) \times CF_t \times \left(1 + \frac{y^{(m)}}{m}\right)^{-tm-2} \times \frac{1}{m}\right]$$

$$= \left(1 + \frac{y^{(m)}}{m}\right)^{-2} \sum_{t\geq 0}\left[t\left(t + \frac{1}{m}\right) \times PV_0(CF_t)\right]$$

The modified convexity is:

$$ModC = \frac{P''\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} = \frac{\left(1+\frac{y^{(m)}}{m}\right)^{-2} \sum_{t \geq 0}\left[t\left(t+\frac{1}{m}\right) \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)}$$

$$= \frac{\left(1+\frac{y^{(m)}}{m}\right)^{-2} \sum_{t \geq 0}\left[t^2 \times PV_0(CF_t)\right] + \sum_{t \geq 0}\left[\frac{t}{m} \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)}$$

$$= \frac{MacC + \frac{MacD}{m}}{\left(1+\frac{y^{(m)}}{m}\right)^2}$$

14.10

Modified Convexity

The modified convexity of an asset can be written in terms of its Macaulay convexity and its Macaulay duration:

$$ModC = \frac{P''\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} = \frac{MacC + \frac{MacD}{m}}{\left(1+\frac{y^{(m)}}{m}\right)^2}$$

The example below uses the Macaulay duration and Macaulay convexity to find the modified convexity of the bond.

Example 14.17

A 2-year bond pays semiannual coupons at an annual rate of 6% per year. The face amount of the bond is \$100. The yield of the bond is 8% compounded semiannually.

a. Calculate the Macaulay duration of the bond.
b. Calculate the Macaulay convexity of the bond.
c. Calculate the modified convexity of the bond.

Solution

a. The price of the bond is:

$$\sum_{t \geq 0} PV_0\left(CF_t\right) = \frac{3}{1.04} + \frac{3}{1.04^2} + \frac{3}{1.04^3} + \frac{103}{1.04^4} = 96.3701$$

The numerator of the formula for Macaulay duration is:

$$\sum_{t \geq 0}\left[t \times PV_0\left(CF_t\right)\right] = \frac{3 \times 0.5}{1.04} + \frac{3 \times 1}{1.04^2} + \frac{3 \times 1.5}{1.04^3} + \frac{103 \times 2}{1.04^4} = 184.3061$$

The Macaulay duration is:

$$MacD = -\frac{P'(r)}{P(r)} = \frac{\sum_{t \geq 0}\left[t \times PV_0\left(CF_t\right)\right]}{\sum_{t \geq 0} PV_0\left(CF_t\right)} = \frac{184.3061}{96.3701} = \mathbf{1.9125}$$

b. The numerator of the formula for Macaulay convexity is:

$$\sum_{t \geq 0}\left[t^2 \times PV_0\left(CF_t\right)\right] = \frac{3 \times 0.5^2}{1.04} + \frac{3 \times 1^2}{1.04^2} + \frac{3 \times 1.5^2}{1.04^3} + \frac{103 \times 2^2}{1.04^4} = 361.6748$$

The Macaulay convexity is:

$$MacC = \frac{P''(r)}{P(r)} = \frac{\sum_{t \geq 0} \left[t^2 \times PV_0(CF_t)\right]}{\sum_{t \geq 0} PV_0(CF_t)} = \frac{361.6748}{96.3701} = \mathbf{3.7530}$$

c. The modified convexity is:

$$ModC = \frac{MacC + \dfrac{MacD}{m}}{\left(1 + \dfrac{y^{(m)}}{m}\right)^2} = \frac{3.7530 + \dfrac{1.9125}{2}}{1.04^2} = \mathbf{4.3539}$$

Suppose that a portfolio consists of k bonds, with prices $P_j(r)$ for $j \in \{1, 2, \cdots, k\}$. The value of the portfolio is:

$$Port = \sum_{j=1}^{k} P_j(r)$$

Contrary to our usual convention, the subscript j indicates which bond we are considering, not the time at which the bond is being valued.

If all of the bonds in the portfolio have the same yield, then the Macaulay convexity of the portfolio is:

$$\begin{aligned} MacC_{Port} &= \frac{1}{Port} \sum_{j=1}^{k} P_j''(r) \\ &= \frac{P_1(r)}{Port} \times \frac{P_1''(r)}{P_1(r)} + \frac{P_2(r)}{Port} \times \frac{P_1''(r)}{P_2(r)} + \cdots + \frac{P_k(r)}{Port} \times \frac{P_1''(r)}{P_k(r)} \\ &= w_1 \times MacC_1 + w_2 \times MacC_2 + \cdots + w_k \times MacC_k \\ &= \sum_{j=1}^{k} w_j \times MacC_j \end{aligned}$$

where:

$$MacC_j = \text{Macaulay convexity of the } j^{\text{th}} \text{ asset}$$

$$w_j = \frac{\text{Present value of } j^{\text{th}} \text{ asset}}{\text{Present value of the portfolio}}$$

The Macaulay convexity of the portfolio is the weighted average of the Macaulay convexities of its component assets, with the weights being the value of each component asset. A similar analysis produces a similar result for modified convexity.

14.11

Portfolio Convexities

The Macaulay and modified convexities of a portfolio are the weighted averages of the convexities of the components:

$$MacC_{Port} = \sum_{j=1}^{k} w_j \times MacC_j$$

$$ModC_{Port} = \sum_{j=1}^{k} w_j \times ModC_j$$

where:

$$MacC_j = \text{Macaulay convexity of the } j^{\text{th}} \text{ asset}$$

$$ModC_j = \text{Modified convexity of the } j^{\text{th}} \text{ asset}$$

$$w_j = \frac{\text{Present value of } j^{\text{th}} \text{ asset}}{\text{Present value of the portfolio}}$$

Example 14.18 A portfolio consists of 3 zero-coupon bonds, each of which matures for \$1,000 in 5, 10, and 15 years, respectively. The yield of the bonds is 8% compounded semiannually.

Calculate the Macaulay convexity of the portfolio.

Solution Since the bonds are zero-coupon bonds, their Macaulay durations and Macaulay convexities are:

$$MacD_5 = 5 \qquad MacC_5 = 5^2 = 25$$

$$MacD_{10} = 10 \qquad MacC_{10} = 10^2 = 100$$

$$MacD_{15} = 15 \qquad MacC_{15} = 15^2 = 225$$

The Macaulay convexity of the portfolio is the weighted average of the Macaulay convexities of the individual bonds:

$$MacC_{Port} = \sum_{j=1}^{k} w_j \times MacC_j$$

$$= \frac{\frac{1{,}000}{1.04^{10}} \times 25 + \frac{1{,}000}{1.04^{20}} \times 100 + \frac{1{,}000}{1.04^{30}} \times 225}{\frac{1{,}000}{1.04^{10}} + \frac{1{,}000}{1.04^{20}} + \frac{1{,}000}{1.04^{30}}}$$

$$= \frac{\frac{1}{1.04^{10}} \times 25 + \frac{1}{1.04^{20}} \times 100 + \frac{1}{1.04^{30}} \times 225}{\frac{1}{1.04^{10}} + \frac{1}{1.04^{20}} + \frac{1}{1.04^{30}}} = \frac{131.8995}{1.4403}$$

$$= \mathbf{91.5797}$$

14.04 Macaulay Duration as an Investment Horizon

Recall from Section 3.02 that the current value of an asset at time t is the sum of the accumulated value of the cash flows occurring prior to time t and the present value of the cash flows occurring on or after time t. Another way to describe the current value at time t is to say that it is the accumulated value of the time-0 current value. Using y to denote the annual effective yield, the expression below shows the functional relationship of the current value with respect to the yield:

$$CV_t(y) = CV_0(y) \times (1+y)^t$$

Since the current value at time 0 is equal to the price of the asset at time 0, we have:

$$CV_t(y) = P(y) \times (1+y)^t$$

Is there a value of t such that the current value at time t is relatively insensitive to changes in y? To answer that question, we set the derivative of the current value equal to 0 and solve for t:

$$\frac{d\left[CV_t(y)\right]}{dy} = 0$$

$$P'(y) \times (1+y)^t + P(y) \times t(1+y)^{t-1} = 0$$

$$P'(y) + P(y) \times t(1+y)^{-1} = 0$$

$$t = -\frac{P'(y)}{P(y)} \times (1+y)$$

$$t = ModD \times (1+y)$$

$$t = MacD$$

The answer to the question above is yes. If an investor holds an asset for a length of time that is equal to the Macaulay duration, then the current value of the asset is relatively unaffected by a small, immediate change in the yield. Making use of the first two terms of the Taylor Series for the current value, we have:

$$CV_{MacD}(y + \Delta y) \approx CV_{MacD}(y) + \Delta y \times \frac{d\left[CV_{MacD}(y)\right]}{dy} = CV_{MacD}(y) + \Delta y \times 0$$

$$= P(y) \times (1+y)^{MacD}$$

Since Δy does not appear to the right of the final equal sign in the expression above, the estimate for the current value does not depend on Δy.

Let's consider how a change in the yield affects the current value.

- Price effect: If t is relatively small, then the current value consists largely of the present value of future cash flows. Since the present value decreases as the yield increases, the derivative of the current value is negative:

  $$\frac{d\left[CV_t(y)\right]}{dy} < 0 \quad \text{for small values of } t$$

- Reinvestment effect: If t is relatively large, then the current value consists largely of the accumulated value of reinvested funds. Since the accumulated value increases as the yield increases, the derivative of the current value is positive:

  $$\frac{d\left[CV_t(y)\right]}{dy} > 0 \quad \text{for large values of } t$$

The Macaulay duration is the value of t that balances the price effect and the reinvestment effect:

$$\frac{d\left[CV_t(y)\right]}{dy} = 0 \quad \text{for } t = MacD$$

14.12

Macaulay Duration as an Investment Horizon

If an investor holds an asset for an interval of time that is equal to the asset's Macaulay duration, then the value of the asset at the end of that time interval is relatively unaffected by small, immediate changes in the yield:

$$CV_{MacD}(y + \Delta y) \approx P(y) \times (1 + y)^{MacD}$$

The initial yield, y, is used to calculate the value of *MacD* in the Key Concept above. The subsequent shift in the yield, $\Delta y^{(m)}$, occurs immediately and remains in place thereafter.

The preceding sentence means that we are assuming that the yield curve is flat, and we are considering a single parallel shift to the yield curve that occurs immediately. Yield curves are discussed in more detail in Section 16.01.

Although the discussion above is based on an assumption that the yield is an annual effective yield, the conclusion that the current value at time t = MacD is relatively unaffected by small, immediate changes in the yield is valid regardless of the compounding of the yield:

$$CV_t\left(y^{(m)}\right) = P\left(y^{(m)}\right) \times \left(1 + \frac{y^{(m)}}{m}\right)^{tm}$$

$$\frac{d\left[CV_t\left(y^{(m)}\right)\right]}{dy^{(m)}} = 0$$

$$P'\left(y^{(m)}\right) \times \left(1 + \frac{y^{(m)}}{m}\right)^{tm} + P\left(y^{(m)}\right) \times tm\left(1 + \frac{y^{(m)}}{m}\right)^{tm-1} = 0$$

$$t = -\frac{P'\left(y^{(m)}\right)}{P\left(y^{(m)}\right)} \times \left(1 + \frac{y^{(m)}}{m}\right)$$

$$t = ModD \times \left(1 + \frac{y^{(m)}}{m}\right)$$

$$t = MacD$$

Therefore, for a small value of $\Delta y^{(m)}$, we have:

$$CV_{MacD}\left(y^{(m)} + \Delta y^{(m)}\right) \approx P\left(y^{(m)}\right) \times \left(1 + \frac{y^{(m)}}{m}\right)^{MacD \times m}$$

Since $\Delta y^{(m)}$ does not appear to the right of the equal sign in the expression above, the estimate for the current value does not depend on $\Delta y^{(m)}$.

Example 14.19

A 3-year bond pays annual coupons at an annual rate of 8% per year. The face amount of the bond is \$100. The annual effective yield of the bond is 8%.

Sue, Karen, and Betsy each buy the bond. Immediately afterward, the annual effective yield increases to 9% and remains at 9%. While they own the bond, Sue, Karen, and Betsy reinvest the cash flows into a cash account earning the 9% yield.

a. Calculate the price and Macaulay duration of the bond when the bond is purchased and the yield is 8%.

b. Calculate the annual effective yield earned by Sue if she holds the bond for 1 year.

c. Calculate the annual effective yield earned by Karen if she holds the bond for 2.7833 years.

d. Calculate the annual effective yield earned by Betsy if she holds the bond for 4 years.

Solution a. Since the yield is equal to the coupon rate, the price of the bond is equal to its par value:

$$\sum_{t\geq 0} PV_0\left(CF_t\right) = \frac{8}{1.08} + \frac{8}{1.08^2} + \frac{108}{1.08^3} = \mathbf{100}$$

The Macaulay duration is:

$$MacD = \ddot{a}_{\overline{3}|0.08} = \frac{1-(1.08)^{-3}}{0.08 / 1.08} = \mathbf{2.7833}$$

b. At the end of 1 year, the value of the first coupon plus the present value of the remaining bond payments is:

$$CV_1 = 8 + \frac{8}{1.09} + \frac{108}{1.09^2} = 106.2409$$

The yield earned by Sue is:

$$\frac{106.2409}{100} - 1 = \mathbf{6.2409\%}$$

c. At the end of 2.7833 years, the accumulated value of the reinvested coupons plus the present value of the remaining bond payments is:

$$CV_{2.7833} = 8 \times 1.09^{(2.7833-1)} + 8 \times 1.09^{(2.7833-2)} + 108 \times 1.09^{(2.7833-3)} = 123.8895$$

The yield earned by Karen is:

$$\left(\frac{123.8895}{100}\right)^{\frac{1}{2.7833}} - 1 = \mathbf{8.0005\%}$$

d. At the end of 4 years, the accumulated value of the reinvested coupons and par value is:

$$CV_4 = 8 \times 1.09^3 + 8 \times 1.09^2 + 108 \times 1.09 = 137.5850$$

The yield earned by Betsy is:

$$\left(\frac{137.5850}{100}\right)^{\frac{1}{4}} - 1 = \mathbf{8.3036\%}$$

The example above shows that the yield earned by holding the bond for a length of time that is equal to the Macaulay duration of 2.7833 is very close to the yield of 8% that was available at the time of purchase. This is because the current value at time 2.7833 is relatively unaffected by the change in the yield. If the yield had remained 8%, then the current value would have been:

$$CV_{2.7833} = 8 \times 1.08^{1.7833} + 8 \times 1.08^{0.7833} + 108 \times 1.08^{-0.2167} = 123.8877$$

The current value of 123.8877 found above is very close to the current value of 123.8895 that was obtained using the new yield of 9%.

14.05 First-Order Macaulay Approximation

The current value at time *t* can be described as the accumulated value of the time-0 price of the asset. If *t* = *MacD* and the yield is $y + \Delta y$, then we have:

$$CV_t(y + \Delta y) = P(y + \Delta y) \times (1 + y + \Delta y)^t$$

As shown in the preceding section, the current value of an asset at time *t*, where *t* = *MacD*, is relatively unchanged by a small, immediate change in the yield of the asset:

$$CV_{MacD}(y + \Delta y) \approx P(y) \times (1 + y)^{MacD}$$

Substituting for the current value allows us to find an estimate for $P(y + \Delta y)$:

$$CV_{MacD}(y + \Delta y) \approx P(y) \times (1 + y)^{MacD}$$
$$P(y + \Delta y) \times (1 + y + \Delta y)^{MacD} \approx P(y) \times (1 + y)^{MacD}$$
$$P(y + \Delta y) \approx P(y) \times \left(\frac{1 + y}{1 + y + \Delta y}\right)^{MacD}$$

Robert Alps, in a study note written for the Society of Actuaries, derives the estimate above, which he refers to at the **first-order Macaulay approximation**. Alps shows that if all of the asset's cash flows are positive, then the first-order Macaulay approximation is at least as good as, and often much better than, the first-order modified (FoMod) approximation.

For convenience, we abbreviate "first-order Macaulay" as FoMac.

14.13

First-Order Macaulay Approximation (FoMac Approximation)

The first-order Macaulay approximation for the new price resulting from a shift in the yield from y to $y + \Delta y$ is:

$$P(y + \Delta y) \approx P(y) \times \left(\frac{1 + y}{1 + y + \Delta y}\right)^{MacD}$$

Where $MacD$ is based on the original yield y.

The Key Concept above is expressed in terms of an annual effective yield. A more general version of the FoMac approximation, expressed in terms of $y^{(m)}$, is shown below:

$$P\left(y^{(m)} + \Delta y^{(m)}\right) \approx P\left(y^{(m)}\right) \times \left(\frac{1 + \dfrac{y^{(m)}}{m}}{1 + \dfrac{y^{(m)} + \Delta y^{(m)}}{m}}\right)^{MacD \times m}$$

The example below demonstrates that if an asset produces a single cash flow, then the FoMod approximation contains some error but the FoMac approximation is exact.

Example 14.20

While the annual effective yield is 8%, Alex purchases a 3-year zero-coupon bond that pays \$100 at time 3.

Immediately after the purchase, the yield falls to 7%.

a. Estimate the new price of the bond using the first-order modified approximation.

b. Estimate the new price of the bond using the first-order Macaulay approximation.

c. Calculate the exact new price.

Solution

a. The Macaulay duration of a zero-coupon bond is equal to the maturity of the bond, so the modified duration is:

$$ModD = \frac{MacD}{1 + y} = \frac{3}{1.08} = 2.7778$$

The FoMod approximation uses the modified duration of 2.7778:

$$P(y + \Delta y) \approx P(y) \times [1 - ModD \times \Delta y] = \frac{100}{1.08^3} \times [1 - 2.7778 \times (-0.01)]$$
$$= 79.3832 \times 1.027778 = \mathbf{81.5883}$$

b. The FoMac approximation uses the Macaulay duration of 3:

$$P(y+\Delta y) \approx P(y) \times \left(\frac{1+y}{1+y+\Delta y}\right)^{MacD} = \frac{100}{1.08^3} \times \left(\frac{1.08}{1.07}\right)^3 = \frac{100}{1.07^3} = \mathbf{81.6298}$$

c. The exact new price is calculated based on the new yield of 7%:

$$P(y+\Delta y) = \frac{100}{1.07^3} = \mathbf{81.6298}$$

When the asset consists of multiple positive cash flows, the FoMac approximation is not exact, but it outperforms the first-order modified duration.

Example 14.21 A 3-year bond pays annual coupons at an annual rate of 8% per year. The face amount of the bond is \$100. The annual effective yield of the bond is 8%.

Immediately afterward, the annual effective yield increases to 9%.

a. Calculate the price, Macaulay duration, and modified duration of the bond when the bond is purchased and the yield is 8%.
b. Estimate the new price of the bond using the first-order modified approximation.
c. Estimate the new price of the bond using the first-order Macaulay approximation.
d. Calculate the exact new price.

Solution a. Since the yield is equal to the coupon rate, the price of the bond is equal to its par value:

$$\sum_{t \geq 0} PV_0(CF_t) = \frac{8}{1.08} + \frac{8}{1.08^2} + \frac{108}{1.08^3} = \mathbf{100}$$

The Macaulay duration is:

$$MacD = \ddot{a}_{\overline{3}|0.08} = \mathbf{2.7833}$$

The modified duration is:

$$ModD = a_{\overline{3}|0.08} = \mathbf{2.5771}$$

b. The FoMod approximation uses the modified duration of 2.5771:

$$P(y+\Delta y) \approx P(y) \times [1 - ModD \times \Delta y] = 100 \times [1 - 2.5771 \times (0.01)]$$
$$= 100 \times 0.9742 = \mathbf{97.4229}$$

c. The FoMac approximation uses the Macaulay duration of 2.7833:

$$P(y+\Delta y) \approx P(y) \times \left(\frac{1+y}{1+y+\Delta y}\right)^{MacD} = 100 \times \left(\frac{1.08}{1.09}\right)^{2.7833} = \mathbf{97.4674}$$

d. The exact new price is calculated based on the new yield of 9%:

$$P(y+\Delta y) = \sum_{t \geq 0} PV_0(CF_t) = \frac{8}{1.09} + \frac{8}{1.09^2} + \frac{108}{1.09^3} = \mathbf{97.4687}$$

14.06 Questions

Question 14.01

A 9-year, 7,000 par value bond with an annual coupon rate of 6.5% paid semiannually is purchased with no premium or discount.

Calculate the Macaulay duration of the bond.

A 6.32 B 6.52 C 6.73 D 6.95 E 7.18

Question 14.02

A portfolio consists of 3 bonds:

Bond 1 has a duration of 8, and its price is 100.

Bond 2 has a duration of 13, and it pays semiannual coupons at an annual rate of 6%. The price of Bond 2 is 75.50.

Bond 3 has a duration of 12, and it matures in 20 years. The price of Bond 3 is 60.41.

Calculate the duration of the portfolio.

A 10.6 B 11.0 C 11.5 D 12.7 E 12.9

Question 14.03

The Macaulay duration of a 3-year bond is X. The bond makes annual coupon payments, and the next coupon payment will be made at the end of one year.

The par value and the redemption value of the bond are both 100. The coupon rate is 6%, and the annual effective yield is 8%.

Six months later, the yield of the bond has not changed, and the new Macaulay duration of the bond is Y.

Calculate $(X - Y)$.

A 0.40 B 0.43 C 0.45 D 0.46 E 0.50

Question 14.04

The Macaulay duration of an n-year bond is 9.8. The bond makes annual coupon payments, and the next coupon payment will be made at the end of one year.

Six months later, the yield of the bond has not changed, and the new Macaulay duration of the bond is X.

Calculate X.

A 8.8 B 9.0 C 9.3 D 9.5 E 9.8

Question 14.05

Consider two annuities that have the same yield and make annual payments.

The first annuity is a level n-year annuity-due, and its Macaulay duration is 3.

The second annuity is a level n-year annuity-immediate, and its Macaulay duration is X.

Calculate X.

A 3.00 B 3.25 C 3.50 D 4.00 E 4.50

Question 14.06

A bond will pay coupons of 80 at the end of each year for 3 years and will pay the par value of 1,000 at the end of the three-year period. The annual effective yield of the bond is 15%.

Calculate the bond's Macaulay duration.

A 2.40 B 2.53 C 2.76 D 2.91 E 3.00

Question 14.07

An asset has cash flows of 50,000 at the end of 2 years, 35,000 at the end of 3 years, and 120,000 at the end of 4 years. The annual effective yield of the asset is 6%.

Calculate the asset's Macaulay duration.

A 2.3 B 2.9 C 3.1 D 3.3 E 3.5

Question 14.08

A seven-year 100 par value bond has 10% annual coupons and an annual effective yield of 7%.

Calculate the Macaulay duration of the bond.

A 3.7 B 4.8 C 5.1 D 5.5 E 5.7

Question 14.09

A nine-year, 6,000 par bond with an annual coupon rate of 4% paid annually sells for 6,000.

Let D_B be the Macaulay duration just before the first coupon is paid.

Let D_A be the Macaulay duration just after the first coupon is paid.

Assume that the yield stays constant at 4%.

Calculate the ratio $\dfrac{D_B}{D_A}$.

A 0.94 B 0.96 C 0.98 D 1.00 E 1.04

Question 14.10

An n-year, par bond with an annual coupon rate of 4% paid annually sells for its par value. You are given that n is greater than 4.

Let D_B be the Macaulay duration just before the 3rd coupon is paid.

Let D_A be the Macaulay duration just after the 3rd coupon is paid.

Assume that the yield stays constant at 4%.

Calculate the ratio $\dfrac{D_B}{D_A}$.

A 0.94 B 0.96 C 0.98 D 1.00 E 1.04

Question 14.11

Consider two annuities that have the same yield and make annual payments.

The first annuity is a level 3-year annuity-due, and its Macaulay duration is 0.96.

The second annuity is a level 4-year annuity-due, and its Macaulay duration is X.

Calculate X.

A 1.1 B 1.3 C 1.4 D 2.0 E 2.4

Question 14.12

A nine-year bond with an annual coupon rate of 6% makes annual coupon payments. The par value of the bond is 1,000, and its redemption value is also 1,000.

Let D_B be the Macaulay duration just before the 4th coupon is paid.

Let D_A be the Macaulay duration just after the 4th coupon is paid.

The yield of the bond stays constant at 4%.

Calculate the ratio $\frac{D_B}{D_A}$.

A 0.94 B 0.95 C 0.96 D 1.00 E 1.06

Question 14.13

An annual coupon bond has a price of 100. The annual effective yield to maturity of the bond is 5%. The derivative of the price of the bond with respect to its yield is −1,000.

Calculate the Macaulay duration of the bond.

A 9.5 B 9.8 C 10.0 D 10.3 E 10.5

Question 14.14

Stock XYZ pays a constant dividend at the end of each year into perpetuity.

Using an annual effective interest rate of 8%, calculate the Macaulay duration of the stock.

A 11.6 B 12.0 C 12.5 D 13.5 E 14.6

Question 14.15

A bond pays annual coupons, and the next coupon will be paid in one year. The price, par value, and redemption value of the bond are all 100. The Macaulay duration of the bond is 6.88 years. The coupon rate of the bond is 7.4%.

Calculate the first-order modified approximation of the price of the bond if the yield of the bond falls to 6.8%.

A 103.84 B 103.94 C 104.13 D 104.94 E 105.13

Question 14.16

Bond A is a 4-year bond that pays annual coupons. The coupon rate is 8%. The bond yields an annual effective rate of 8%. The modified duration of Bond A with respect to its 8% yield is K.

Bond B pays semiannual coupons and yields 8% convertible semiannually. The modified duration of Bond B with respect to its 8% yield is K.

Calculate the Macaulay duration of Bond B.

A 3.0 B 3.1 C 3.2 D 3.3 E 3.4

Question 14.17

Stock XYZ pays dividends at the end of each year into perpetuity. The dividend increases by 1% each year.

Using an annual effective interest rate of 4%, calculate the Macaulay duration of the stock.

A 32.1 B 33.3 C 34.7 D 36.1 E 37.5

Question 14.18

A perpetuity pays 1 at the beginning of each year. The perpetuity's Macaulay duration is 28 years. The yield of the perpetuity is expressed as an annual effective interest rate.

Calculate the modified duration of the perpetuity.

A 27.0 B 27.2 C 27.4 D 27.6 E 27.8

Question 14.19

A par value bond has a price that is equal to its par value of 100. The bond pays semiannual coupons. The yield-to-maturity of the bond is 5% convertible semiannually. The derivative of the price of the bond with respect to its yield is −1,000.

Calculate the number of years until the bond matures.

A 12.0 B 12.5 C 13.0 D 14.0 E 14.5

Question 14.20

A 20-year bond with a par value of 1,000 pays annual coupons at a rate of 5%. You are given:

- The modified duration of the bond is 11.904.
- The modified convexity of the bond is 197.238.

The annual effective yield of the bond is 6%.

Use the modified duration and modified convexity to estimate the percentage change in the price of the bond if the yield decreases to 5%.

A 9.93% B 10.92% C 11.90% D 12.89% E 13.88%

Question 14.21

A 20-year bond with a par value of 1,000 pays annual coupons at a rate of 5%. You are given:

- The price of the bond is 885.30.
- The modified duration of the bond is 11.904.
- The modified convexity of the bond is 197.238.

The annual effective yield of the bond is 6%.

An analyst uses the modified duration and the modified convexity to estimate the percentage change in the price of the bond if the yield decreases to 5%. The resulting estimate for the new price is X.

If the yield decreases to 5%, the actual new price will be Y.

Calculate $X - Y$.

A −9.31 B −0.58 C 0.00 D 0.58 E 9.31

Question 14.22

Ann receives a substantial sum for the sale of her business.

She invests half of the sum in a 10-year zero-coupon bond, and she invests the other half in a 20-year zero-coupon bond.

Calculate the Macaulay convexity of her investment.

A 225 B 235 C 240 D 250 E 260

Question 14.23

Mike purchases two zero-coupon bonds.

- The first bond matures in 10 years for 895.42.
- The second bond matures in 20 years for 1,603.57.

The annual effective yield is 6.0%.

Calculate the modified convexity of Mike's portfolio.

A 209.15 B 221.70 C 235.85 D 225.00 E 250.00

Question 14.24

Stock XYZ pays a constant dividend at the end of each year into perpetuity.

Using an annual effective interest rate of 8%, calculate the modified convexity of the stock.

A 150.00 B 156.25 C 168.75 D 182.25 E 312.50

Question 14.25

Stock XYZ pays a constant dividend at the end of each year into perpetuity.

Using an annual effective interest rate of 8%, calculate the Macaulay convexity of the stock.

A 324 B 325 C 351 D 352 E 378

Question 14.26

The yield of a 25-year bond is 8%, and the Macaulay duration of the bond is 14.

If the yield immediately decreases by 0.40% (40 basis points), then the bond's approximate percentage change in price is X, based on a first-order Macaulay approximation.

Calculate X.

A 4.93% B 5.04% C 5.19% D 5.33% E 5.77%

Question 14.27

An n-year 1,000 bond is purchased at par to yield an annual effective rate of 7.4%. The Macaulay duration of the bond is 8.776.

Calculate the first-order Macaulay approximation for the price of the bond if the interest rate falls to 7.0%.

A 1,028.09 B 1,029.70 C 1,033.29 D 1,035.10 E 1,035.80

Question 14.28

A bond has a price of 850.46 and a modified duration of 9, calculated using an annual effective interest rate of 7.4%.

The first-order Macaulay approximation, at an interest rate of 8.0%, for the price of the bond is E_{Mac}.

The first-order modified approximation, at an interest rate of 8.0%, for the price of the bond is E_{Mod}.

Calculate $E_{Mac} - E_{Mod}$.

A 1.34 B 4.09 C 4.33 D 6.18 E 6.77

Question 14.29

Kris has a portfolio that consists of 2 bonds:

- Bond A has a price of 70,000 and a Macaulay duration of 5.4.
- Bond B has a price of 30,000 and a Macaulay duration of 11.9.

The prices and durations of both bonds were calculated using an annual effective interest rate of 6.5%.

Using the first-order Macaulay approximation, Kris estimates the value of the portfolio to be 96,000 if the interest rate changes to *i*.

Calculate *i*.

A 5.91% B 6.73% C 6.94% D 7.09% E 7.13%

Question 14.30

At an annual effective interest rate of 7.4%, the present value of an annuity-immediate is 59,776.39 and its modified duration is *ModD*.

Using the first-order Macaulay approximation to estimate the present value of the annuity-immediate results in an estimate of 57,021.04 if the interest rate is 8.2%.

Calculate *ModD*.

A 5.88 B 5.92 C 6.18 D 6.36 E 6.83

Chapter 15: Asset-Liability Matching

15.01 Interest Rate Risk

Assets produce cash inflows, which are called asset cash flows. Liabilities produce cash outflows, which are called liability cash flows. The surplus of the present value of a firm's asset cash flows over the present value of the firm's liability cash flows is often simply called the firm's **surplus**:

$$S(y) = PV_A - PV_L$$

where:

$S(y)$ = Surplus when the yield is y

PV_A = Present value of asset cash flows

PV_L = Present value of liability cash flows

Example 15.01 A firm must pay \$100 at the end of 5 years. The firm has a 1-year zero-coupon bond that matures for \$66.15 and a 20-year zero-coupon bond that matures for \$31.72.

Calculate the surplus of the firm, using an annual effective interest rate of 8% to value the cash flows.

Solution The present value of the liability cash flow is:

$$PV_L = \frac{100}{1.08^5} = 68.06$$

The present value of the asset cash flows is:

$$PV_A = \frac{66.15}{1.08} + \frac{31.72}{1.08^{20}} = 68.06$$

The surplus of the firm is:

$$S(0.08) = PV_A - PV_L = 68.06 - 68.06 = \mathbf{0.00}$$

If the yield decreases, then the firm in the example above faces **reinvestment risk** because the 1-year bond matures prior to the liability payment and must be reinvested at the new, lower yield. Reinvestment risk is the risk of earning a relatively low interest rate when investing future cash flows.

Example 15.02 A firm must pay \$100 at the end of 5 years. The firm has a 1-year zero-coupon bond that matures for \$66.15 and a 20-year zero-coupon bond that matures for \$31.72.

Both bonds were purchased when the annual effective interest rate was 8%.

Immediately after the firm purchases the bonds, the annual effective interest rate falls to 6%. Calculate the new surplus of the firm.

Solution The present value of the liability cash flow is higher than before because the new interest rate of 6% is lower than the previous interest rate of 8%:

$$PV_L = \frac{100}{1.06^5} = 74.73$$

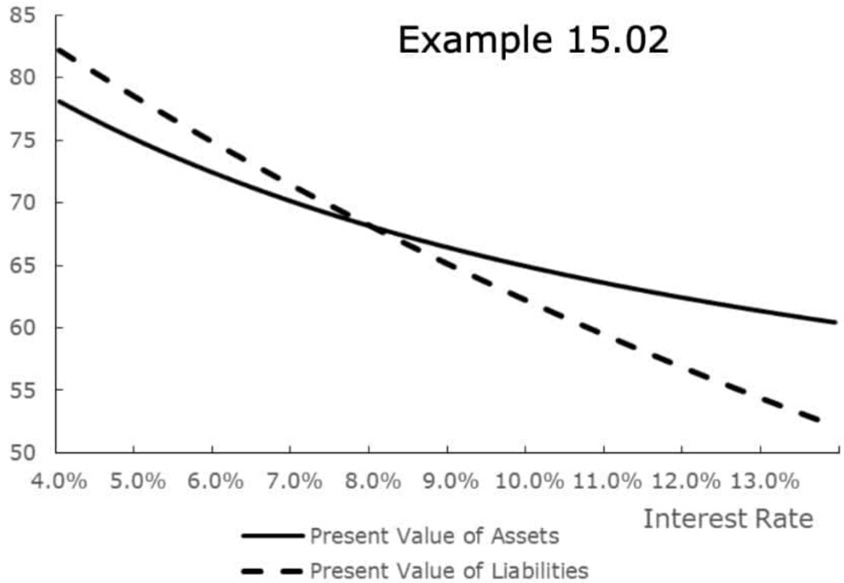
The present value of the asset cash flows is also higher before because the new interest rate of 6% is lower than the previous interest rate of 8%:

$$PV_A = \frac{66.15}{1.06} + \frac{31.72}{1.06^{20}} = 72.30$$

The surplus of the firm is now negative because the present value of the liabilities increased more than the present value of the assets:

$$S(0.06) = PV_A - PV_L = 72.30 - 74.73 = -\mathbf{2.43}$$

The graph below shows that when the interest rate decreases, the value of the asset cash flows described in the example above increases by less than the value of the liability cash flow.



To reduce reinvestment risk, the firm in the above example could reduce the amount invested in the shorter bond and increase the amount invested in the longer bond. This strategy exposes the firm to **price risk** though. Price risk is the risk that assets to be sold in the future could sell for a relatively low price if the yield on the assets increases.

Example 15.03

A firm must pay \$100 at the end of 5 years. Observing the potential for reinvestment risk, the firm purchases less of the 1-year bond and more of the 20-year bond than shown in the previous example. The firm purchases a 1-year zero-coupon bond that matures for \$36.75 and a 20-year zero-coupon bond that matures for \$158.61. Both bonds were purchased when the annual effective interest rate was 8%.

a. Calculate the surplus of the firm when the annual effective interest rate is 8%.

b. Calculate the surplus of the firm if the annual effective interest rate immediately increases to 10%.

Solution

a. The present value of the liability cash flow is:

$$PV_L = \frac{100}{1.08^5} = 68.06$$

The present value of the asset cash flows is:

$$PV_A = \frac{36.75}{1.08} + \frac{158.61}{1.08^{20}} = 68.06$$

The surplus of the firm is:

$$S(0.08) = PV_A - PV_L = 68.06 - 68.06 = \mathbf{0.00}$$

b. The present value of the liability cash flow is lower than before because the new interest rate of 10% is higher than the previous interest rate of 8%:

$$PV_L = \frac{100}{1.10^5} = 62.09$$

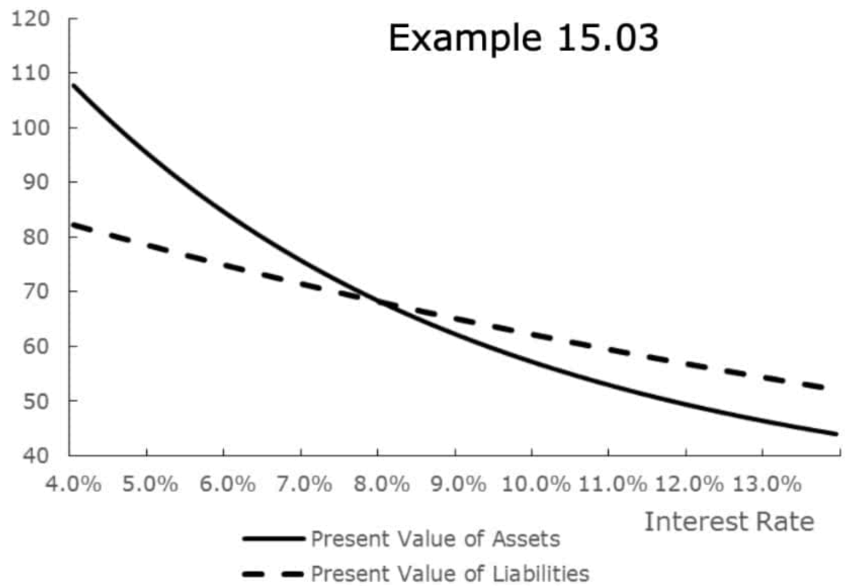
The present value of the asset cash flows is also lower than before because the new interest rate of 10% is higher than the previous interest rate of 8%:

$$PV_A = \frac{36.75}{1.10} + \frac{158.61}{1.10^{20}} = 56.99$$

The surplus of the firm is negative because the present value of the liabilities increased more than the present value of the assets:

$$S(0.10) = PV_A - PV_L = 56.99 - 62.09 = \mathbf{-5.11}$$

The graph below shows that when the interest rate increases, the value of the asset cash flows described in the example above falls below the value of the liability cash flow.



Reinvestment risk and price risk are examples of **interest rate risk**. As interest rates (or yields) change, the surplus of the firm changes.

15.02 Cash Flow Matching

One way to avoid interest rate risk is to match the asset cash flows with the liability cash flows exactly. **Cash flow matching** eliminates reinvestment risk because no funds are reinvested after the initial investment. Cash flow matching avoids price risk because it is not necessary to sell any assets to fund the liabilities. Cash flow matching is also known as **dedication** because it dedicates each asset cash flow to offset a specified liability cash flow.

An easy way to do this is to purchase zero-coupon bonds that provide cash flows that match the liability cash flows exactly. The example below shows that when a position is cash flow matched, the surplus does not change as the interest rate changes.

Example 15.04

A firm must pay $100 at the end of 5 years. The firm implements a cash flow matching strategy by purchasing a zero-coupon bond that matures for $100 at the end of 5 years.

The annual effective interest rate is 8%.

a. Calculate the surplus of the firm when the annual effective interest rate is 8%.
b. Calculate the surplus of the firm if the annual effective interest rate immediately decreases to 6%.
c. Calculate the surplus of the firm if the annual effective interest rate immediately increases to 10%.

Solution

a. The present value of the liability cash flow is:

$$PV_L = \frac{100}{1.08^5} = 68.06$$

The present value of the asset cash flows is:

$$PV_A = \frac{100}{1.08^5} = 68.06$$

The surplus of the firm is:

$$S(0.08) = PV_A - PV_L = 68.06 - 68.06 = \mathbf{0.00}$$

b. Although the present value of both the liability and the asset increase when the interest rate falls to 6%, they remain equal, so the surplus remains zero:

$$PV_L = \frac{100}{1.06^5} = 74.73$$

$$PV_A = \frac{100}{1.06^5} = 74.73$$

$$S(0.06) = PV_A - PV_L = 74.73 - 74.73 = \mathbf{0.00}$$

c. Although the present value of both the liability and the asset decrease when the interest rate increases to 10%, they remain equal, so the surplus remains zero:

$$PV_L = \frac{100}{1.10^5} = 62.09$$

$$PV_A = \frac{100}{1.10^5} = 62.09$$

$$S(0.10) = PV_A - PV_L = 62.09 - 62.09 = \mathbf{0.00}$$

Zero-coupon bonds are not always available, and it may be necessary to use coupon-paying bonds to match the liability cash flows. To determine how much of each bond to purchase, we begin by matching the liability cash flow that is furthest out. We then use the resulting asset cash flows to offset the liability cash flows, thereby obtaining the net liability cash flows remaining. The net liability cash flow that is furthest out is addressed next by purchasing another bond. This bond's cash flows are also netted against the remaining liability payments to obtain the new net liability payments. This continues until the net liability payments are zero.

Example 15.05

A firm must pay \$414 at the end of 1 year, \$3,144 at the end of 2 years, \$24 at the end of 3 years, and \$824 at the end of 4 years.

Three noncallable bonds are available for the firm to purchase. Each bond has a par value of \$1,000 and pays coupons annually:

- 1-year bond with a coupon rate of 8%
- 2-year bond with a coupon rate of 4%
- 4-year bond with a coupon rate of 3%

All of the bonds have an annual effective yield of 5%.

Calculate the cost to cash flow match the liability payments.

Solution

The final cash flow of \$824 can only be matched by the 4-year bond. The payment of one of the 4-year bonds at the end of 4 years is \$1,030, so the quantity of 4-year bonds to purchase is:

$$Q_4 = \frac{824}{1{,}030} = 0.80$$

Purchasing 0.80 of the 4-year bonds results in coupon payments of \$24 and a final payment of \$824. The net remaining liability cash flows are shown in the final row below:

Year	1	2	3	4
Liability Cash Flow	414.00	3,144.00	24.00	824.00
Asset Cash Flow	24.00	24.00	24.00	824.00
Net Liability Cash Flow	390.00	3,120.00	0.00	0.00

The net payment at the end of 2 years is \$3,120, and the payment of one of the 2-year bonds at the end of 2 years is \$1,040, so the quantity of 2-year bonds to purchase is:

$$Q_2 = \frac{3{,}120}{1{,}040} = 3$$

Purchasing 3 of the 2-year bonds results in coupon payments of \$120 and a final payment of \$3,120. The new net remaining cash flows are shown in the final row below:

Net Liability Cash Flow	390.00	3,120.00
Asset Cash Flow	120.00	3,120.00
New Net Liability Cash Flow	270.00	0.00

The final net payment at the end of 1 year is \$270, and the payment of one of the 1-year bonds is \$1,080, so the quantity of 1-year bonds to purchase is:

$$Q_1 = \frac{270}{1{,}080} = 0.25$$

Purchasing 0.25 of the 1-year bonds results in a payment of \$270 at time 1. This brings the final net liability cash flow to zero:

New Net Liability Cash Flow	270.00
Asset Cash Flow	270.00
Final Net Liability Cash Flow	0.00

Let's find the price of each of the bonds:

$$\text{Price of 1-year bond} = \frac{1{,}080}{1.05} = 1{,}028.57$$

$$\text{Price of 2-year bond} = \frac{40}{1.05} + \frac{1{,}040}{1.05^2} = 981.41$$

$$\text{Price of 4-year bond} = \frac{30}{1.05} + \frac{30}{1.05^2} + \frac{30}{1.05^3} + \frac{1{,}030}{1.05^4} = 929.08$$

Multiplying the quantity of each bond to be purchased by the price of each bond gives us the cost to cash flow match the liability payments:

$$0.25 \times 1{,}028.57 + 3 \times 981.41 + 0.8 \times 929.08 = \mathbf{3{,}944.63}$$

Since the asset cash flows match the liability cash flows, and since the assets all have the same yield, we can find the cost of the assets in the example above by simply discounting the liability cash flows at the yield:

$$\frac{414}{1.05} + \frac{3{,}144}{1.05^2} + \frac{24}{1.05^3} + \frac{824}{1.05^4} = \mathbf{3{,}944.63}$$

15.03 Redington Immunization

Cash flow matching is not always a viable option. The bonds needed to exactly match the liability cash flows may not be available, or they may have too high of a price.

Let's reconsider the risks facing a firm that is not cash flow matched.

- Yield falls. If the yield falls, then the firm encounters reinvestment risk because the funds that must be reinvested may not earn enough interest. On the other hand, a lower yield can also benefit the firm because a lower yield increases the price of assets that must be sold prior to maturity.

- Yield increases. If the yield increases, then the firm encounters price risk because the assets that must be sold might bring too low of a price. On the other hand, a higher yield can also benefit the firm because a higher yield increases the interest earned on the funds that must be reinvested.

Immunization, which was introduced by British actuary Frank M. Redington in 1952, is a way of balancing the reinvestment risk, which is the risk that the yield falls, against the price risk, which is the risk that the yield increases. Immunization is accomplished by setting the duration of the assets equal to the duration of the liabilities.

$$Dur_A = Dur_L$$

Setting the Macaulay durations equal to one another is equivalent to setting the modified durations equal to one another, so we use Dur here to indicate either Macaulay durations or modified durations:

$$MacD_A = MacD_L \qquad \Leftrightarrow \qquad ModD_A = ModD_L$$

Redington immunization, sometimes called classical immunization, is based on an assumption and two rules:

Assumption: The present value of the assets is equal to the present value of the liabilities:

$$PV_A = PV_L$$

Rule 1: The duration of the assets is equal to the duration of the liabilities:

$$Dur_A = Dur_L$$

Rule 2: The convexity of the assets is greater than the convexity of the liabilities:

$$Conv_A > Conv_L$$

As with duration, it doesn't matter whether we use Macaulay convexity or modified convexity, as long as we are consistent. Therefore we use Conv to refer to either Macaulay convexity or modified convexity:

$$MacC_A > MacC_L \qquad \Leftrightarrow \qquad ModC_A > ModC_L$$

The assumption and the two rules are often summarized as three conditions for Redington immunization.

15.01

Redington Immunization

A position that meets the conditions for Redington immunization has positive surplus if the yield changes by a small amount. The conditions for Redington immunization are:

1. The present value of the assets is equal to the present value of the liabilities:

$$PV_A = PV_L$$

2. The duration of the assets is equal to the duration of the liabilities:

$$Dur_A = Dur_L$$

3. The convexity of the assets is greater than the convexity of the liabilities:

$$Conv_A > Conv_L$$

Let's consider the implications of the three conditions:

1. The initial surplus is zero:

$$PV_A = PV_L$$
$$PV_A - PV_L = 0$$
$$S(y) = 0$$

2. The initial duration of the surplus is zero:

$$Dur_A = Dur_L$$

$$-\frac{d(PV_A)}{dy} \times \frac{1}{PV_A} = -\frac{d(PV_L)}{dy} \times \frac{1}{PV_L}$$

$$\frac{d(PV_A)}{dy} = \frac{d(PV_L)}{dy}$$

$$\frac{d(PV_A)}{dy} - \frac{d(PV_L)}{dy} = 0$$

$$\frac{dS(y)}{dy} = 0$$

$$S'(y) = 0$$

3. The initial convexity of the surplus is positive:

$$Conv_A > Conv_L$$

$$\frac{d^2(PV_A)}{dy^2} \times \frac{1}{PV_A} > \frac{d^2(PV_L)}{dy^2} \times \frac{1}{PV_L}$$

$$\frac{d^2(PV_A)}{dy^2} - \frac{d^2(PV_L)}{dy^2} > 0$$

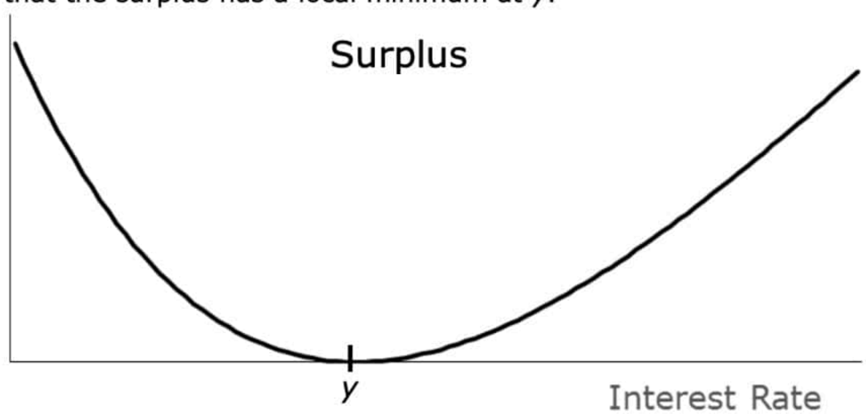
$$\frac{d^2S(y)}{dy^2} > 0$$

$$S''(y) > 0$$

To summarize, we have:

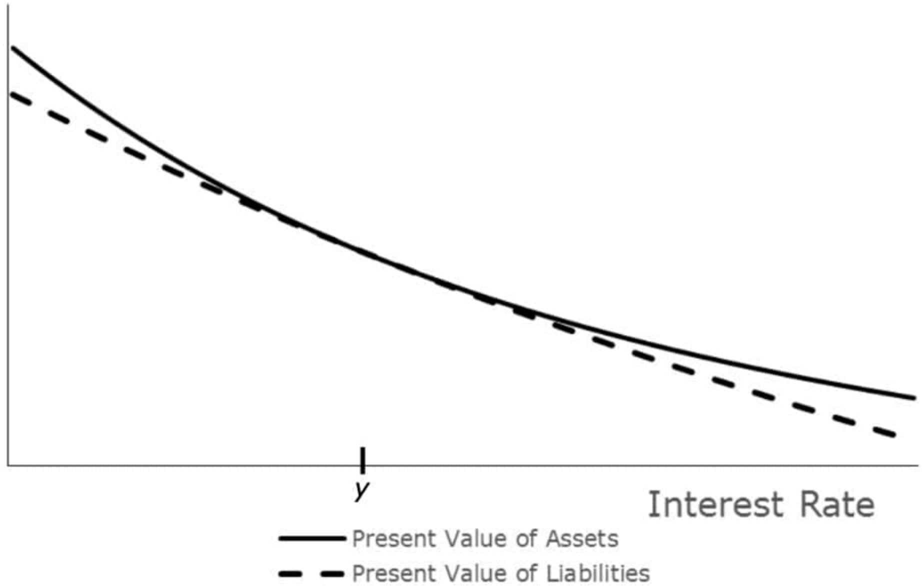
1. $S(y) = 0$
2. $S'(y) = 0$
3. $S''(y) > 0$

This implies that the surplus has a local minimum at y.



Equivalently, we can say that for a small change in y, the present value of the assets increases by more than the present value of the liabilities.

That is, if y decreases by a small amount, then the increase in the present value of the assets is greater than the increase in the present value of the liabilities. If y increases by a small amount, then the decrease in the present value of the assets is smaller than the decrease in the present value of the liabilities.



The surplus seems to change by more in the first graph than in the second graph. That is because the vertical axes of the two graphs above are scaled differently.

Example 15.06

A firm must pay \$100 at the end of 5 years. The firm has 1-year zero-coupon bonds and 20-year zero-coupon bonds available for purchase. The annual effective yield on all assets and liabilities is 8%. The firm implements a Redington immunization strategy.

a. Calculate the amount invested in the 1-year bond and the amount invested in the 20-year bond.

b. Calculate the maturity values of the 1-year bond and the 20-year bond purchased by the firm.

Solution

a. The present value, Macaulay duration, and Macaulay convexity of the liability are:

$$PV_L = \frac{100}{1.08^5} = 68.06 \qquad MacDur_L = 5 \qquad MacC_L = 5^2 = 25$$

To satisfy condition #1, it will be necessary to purchase assets with a value that is equal to the present value of the liabilities, so:

$$PV_A = 68.06$$

To satisfy condition #2, it will be necessary to allocate the invested funds between the 1-year bond and the 20-year bond in such a way that the duration of the asset portfolio is equal to 5. Let x be the amount invested in the 1-year bond:

$$MacDur_1 \times x + MacDur_{20} \times (1 - x) = MacDur_L$$
$$1 \times x + 20 \times (1 - x) = 5$$
$$19x = 15$$
$$x = 0.7895$$

The Macaulay convexity of the asset portfolio is:

$$MacC_1 \times x + MacC_{20} \times (1 - x) = 1^2 \times 0.7895 + 20^2 \times (1 - 0.7895) = 85$$

Since the Macaulay convexity of the asset portfolio is greater than the Macaulay convexity of the liability (i.e., 85 > 25), condition #3 for Redington immunization is satisfied.

The amounts to be invested in the 1-year and the 20-year bonds are:

$$PV_1 = x \times PV_A = 0.7895 \times 68.06 = \mathbf{53.73}$$

$$PV_{20} = (1 - x) \times PV_A = (1 - 0.7895) \times 68.06 = \mathbf{14.33}$$

b. The 1-year and 20-year bonds mature for the following face amounts:

$$F_1 = 0.7895 \times 68.06 \times 1.08 = \mathbf{58.03}$$

$$F_{20} = (1 - 0.7895) \times 68.06 \times 1.08^{20} = \mathbf{66.78}$$

The next example considers the possibility that the yield falls immediately after the portfolio is immunized as described in the example above.

Example 15.07

A firm must pay \$100 at the end of 5 years. The firm has 1-year zero-coupon bonds and 20-year zero-coupon bonds available for purchase. The annual effective yield on all assets and liabilities is 8%. The firm implements a Redington immunization strategy.

a. Suppose that the annual effective yield falls to 6% immediately after the position is immunized. Convert the change in the annual effective yield into a change in the continuously compounded yield, and then use that change with the Macaulay durations and the Macaulay convexities to estimate the new surplus.

b. Calculate the actual value of the surplus if the annual effective yield falls to 6% immediately after the position is immunized.

Solution

a. In the preceding example, we found the following values for the position that has been classically immunized:

$$PV_A = PV_L = 68.06 \qquad MacDur_A = MacDur_L = 5 \qquad MacC_A = 85 \qquad MacC_L = 25$$

The Macaulay duration and convexity are based on the change in the continuously compounded yield. A 200 basis point shift down in the annual effective yield is equivalent to a 187 basis point shift in the continuously compounded yield:

$$\ln(1.06) - \ln(1.08) = -0.0187$$

The estimate for the new value of the assets after a 187 basis point drop in the continuously compounded yield is found below:

$$\%\Delta P \simeq -MacDur \times \Delta y^{(\infty)} + 0.5 \times MacC \times \left(\Delta y^{(\infty)}\right)^2$$

$$= -5 \times (-0.0187) + 0.5 \times 85 \times (-0.0187)^2 = 0.1083$$

$$PV_A(0.06) \simeq 68.06(1 + 0.1083) = 75.43$$

The estimate for the new value of the liability after a 187 basis point drop in the continuously compounded yield is found below:

$$\%\Delta P \simeq -MacDur \times \Delta y^{(\infty)} + 0.5 \times MacC \times \left(\Delta y^{(\infty)}\right)^2$$

$$= -5 \times (-0.0187) + 0.5 \times 25 \times (-0.0187)^2 = 0.0978$$

$$PV_L(0.06) \simeq 68.06(1 + 0.0978) = 74.72$$

The estimate for the new value of the surplus is:

$$S(0.06) \simeq PV_A(0.06) - PV_L(0.06) = 75.43 - 74.72 = \mathbf{0.71}$$

b. The face amounts of the 1-year and the 20-year bonds are:

$$F_1 = 0.7895 \times 68.06 \times 1.08 = 58.03$$

$$F_{20} = (1 - 0.7895) \times 68.06 \times 1.08^{20} = 66.78$$

The actual value of the surplus after the yield falls to 6% is:

$$S(0.06) = PV_A(0.06) - PV_L(0.06) = \frac{58.03}{1.06} + \frac{66.78}{1.06^{20}} - \frac{100}{1.06^5} = \mathbf{0.84}$$

The next example considers the possibility that the yield increases.

Example 15.08

A firm must pay \$100 at the end of 5 years. The firm has 1-year zero-coupon bonds and 20-year zero-coupon bonds available for purchase. The annual effective yield on all assets and liabilities is 8%. The firm implements a Redington immunization strategy.

a. Suppose that the annual effective yield rises to 10% immediately after the position is immunized. Convert the change in the annual effective yield into a change in the continuously compounded yield, and then use that change with the Macaulay durations and the Macaulay convexities to estimate the new surplus.

b. Calculate the actual value of the surplus if the annual effective yield rises to 10% immediately after the position is immunized.

Solution

a. Earlier, we found the following values for the position that has been classically immunized:

$$PV_A = PV_L = 68.06 \qquad MacD_A = MacD_L = 5 \qquad MacC_A = 85 \qquad MacC_L = 25$$

The Macaulay duration and convexity are based on the change in the continuously compounded yield. A 200 basis point shift up in the annual effective yield is equivalent to a 187 basis point shift in the continuously compounded yield:

$$\ln(1.10) - \ln(1.08) = 0.0183$$

The estimate for the new value of the assets after a 183 basis point increase in the continuously compounded yield is found below:

$$\%\Delta P \approx -MacD \times \Delta y^{(\infty)} + 0.5 \times MacC \times \left(\Delta y^{(\infty)}\right)^2$$

$$= -5 \times (0.183) + 0.5 \times 85 \times (0.0183)^2 = -0.077$$

$$PV_A(0.10) \approx 68.06(1 - 0.077) = 62.79$$

The estimate for the new value of the liability after a 183 basis point increase in the continuously compounded yield is found below:

$$\%\Delta P \approx -MacD \times \Delta y^{(\infty)} + 0.5 \times MacC \times \left(\Delta y^{(\infty)}\right)^2$$

$$= -5 \times (0.0183) + 0.5 \times 25 \times (0.0183)^2 = -0.088$$

$$PV_L(0.10) \approx 68.06(1 - 0.088) = 62.10$$

The estimate for the new value of the surplus is:

$$S(0.10) \approx PV_A(0.10) - PV_L(0.10) = 62.79 - 62.10 = \mathbf{0.69}$$

b. The face amounts of the 1-year and the 20-year bonds are:

$$F_1 = 0.7895 \times 68.06 \times 1.08 = 58.03$$

$$F_{20} = (1 - 0.7895) \times 68.06 \times 1.08^{20} = 66.78$$

The actual value of the surplus after the yield rises to 10% is:

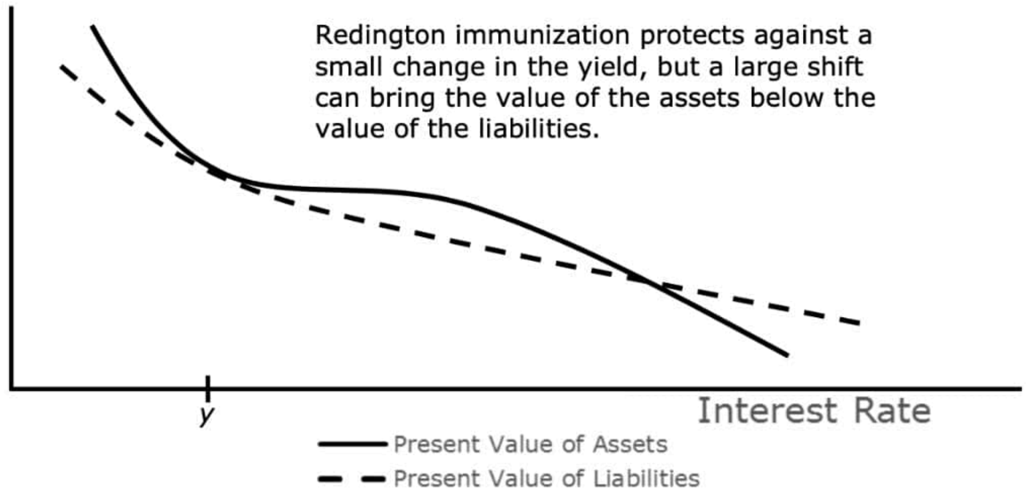
$$S(0.10) = PV_A(0.10) - PV_L(0.10) = \frac{58.03}{1.10} + \frac{66.78}{1.10^{20}} - \frac{100}{1.10^5} = \mathbf{0.59}$$

The values, durations, and convexities of the assets and liabilities change over time, so it is necessary to rebalance the firm's position as time passes. This rebalancing is usually accomplished by buying and selling assets, although it can be accomplished by adjusting the liability position as well.

Redington immunization doesn't necessarily guarantee that the surplus of a portfolio will remain nonzero.

A portfolio that meets the conditions of Redington immunization still faces the following risks:

- Large shifts in the yield. The graphs above showed that for a small change in the yield, the surplus becomes positive. Consider the possibility, as illustrated below, that for a large shift in the yield, the surplus could become negative.



- Rebalancing. The duration and convexity of the asset portfolio changes over time, and the duration and convexity of the liability portfolio also change over time. These changes do not necessarily occur at the same rate for the assets and the liabilities, however, so frequent rebalancing is required to satisfy the conditions for Redington immunization.
- Credit risk. The asset portfolio could contain risky assets, such as corporate bonds, that could potentially default. This text does not consider asset risk, so we don't evaluate it here, but we should be aware of its existence.
- Shape of the yield curve. Redington immunization is based on two assumptions:
    - The yield curve is flat, which means that all assets and liabilities have the same yield at any point in time.
    - Shifts in the yield curve are parallel, which means that the new yield curve is parallel to the original yield curve.

  If either of these two assumptions is violated, then the surplus may become negative.
- Interest-sensitive cash flows. Both the assets and the liabilities could potentially change in response to changes in interest rates. For example, callable bonds are likely to be called when interest rates fall. For this reason, more advanced techniques must be utilized when dealing with interest-sensitive cash flows.

Despite these concerns, Redington immunization remains useful for mitigating the risk that the surplus of a firm will decline due to changes in the yields.

15.04 Full Immunization

Redington immunization protects the surplus from a small shift in the yield, but a large shift in the yield can still result in a negative surplus. **Full immunization**, unlike Redington immunization, protects the surplus from both small and large shifts in the yield. Full immunization considers each liability cash flow individually.

An individual liability cash flow is fully immunized if the position is Redington immunized and one additional, fourth, condition is met.

This gives us the following conditions for full immunization:

1. The present value of the assets is equal to the present value of the liability:

$$PV_A = PV_L$$

2. The duration of the assets is equal to the duration of the liability:

$$Dur_A = Dur_L$$

3. The convexity of the assets is greater than the convexity of the liability:

$$Conv_A > Conv_L$$

4. Asset cash flows occur both before and after the liability cash flow.

The proof that full immunization protects the surplus from any shift in the yield is lengthy, and it appears in the appendix at the end of this chapter.

The four conditions above have the benefit of making it apparent that full immunization is a subset of Redington immunization, but we could instead define full immunization by changing the third condition of Redington immunization. Suppose that the first and second conditions of Redington immunization are met. In that case, if the asset cash flows come before and after the liability cash flow, then the third condition of Redington immunization is certain to be met as well. Since the asset cash flows come before and after the liability cash flow, the dispersion of the assets is greater than zero, and the dispersion of the single liability cash flow is zero:

$$Dispersion_A > 0$$

$$Dispersion_A > Dispersion_L$$

$$MacD_A^2 + Dispersion_A > MacD_L^2 + Dispersion_L$$

$$MacC_A > MacC_L$$

Therefore the conditions for full immunization can be met by replacing the third condition of Redington immunization by:

3. Asset cash flows occur both before and after the liability cash flow.

15.02

Full Immunization for a Single Liability Cash Flow

An individual liability cash flow is fully immunized if the position satisfies the following conditions:

1. The present value of the assets is equal to the present value of the liability:

$$PV_A = PV_L$$

2. The duration of the assets is equal to the duration of the liability:

$$Dur_A = Dur_L$$

3. Asset cash flows occur both before and after the liability cash flow.

Even though full immunization addresses the first bullet point in Section 15.03, the other bullet points remain valid concerns for a fully immunized portfolio.

Although the definition of full immunization in the Key Concept above applies to only one liability cash flow, it is possible to fully immunize multiple liability cash flows by applying the definition above to each liability cash flow.

Example 15.09

A firm must pay \$100 at the end of 5 years. The annual effective yield on all assets and liabilities is 8%. The firm purchases a 1-year zero-coupon bond that matures for \$58.03 and a 20-year zero-coupon bond that matures for \$66.78.

a. Has the firm satisfied the conditions for full immunization?

b. Has the firm satisfied the conditions for Redington immunization?

Solution

a. The present value, Macaulay duration, and Macaulay convexity of the liability are:

$$PV_L = \frac{100}{1.08^5} = 68.06 \qquad MacD_L = 5 \qquad MacC_L = 5^2 = 25$$

To satisfy condition #1, the present value of the assets must be equal to the present value of the liability. As shown below, condition #1 is satisfied:

$$PV_A = \frac{58.03}{1.08} + \frac{66.78}{1.08^{20}} = 68.06$$

To satisfy condition #2, the duration of the assets must be equal to the duration of the liability. As shown below, condition #2 is satisfied:

$$MacD_{Assets} = \sum_{j=1}^{k} w_j \times MacD_j = \frac{\frac{58.03}{1.08} \times 1 + \frac{66.78}{1.08^{20}} \times 20}{\frac{58.03}{1.08} + \frac{66.78}{1.08^{20}}} = 5$$

To satisfy condition #3, asset cash flows must come before and after the liability cash flows. The asset cash flow from the 1-year bond comes before the time-5 liability cash flow, and the asset cash flow from the 20-year bond comes after it, so condition #3 is satisfied.

The firm has satisfied the conditions for full immunization, so the answer is **yes**.

b. Full immunization is a subset of Redington immunization, so the firm has also satisfied the conditions of Redington immunization, and the answer is **yes**.

If liabilities are cash flow matched with a dedicated portfolio, is the position fully immunized? Technically, no, because in a cash flow matched position, the convexity of the assets is equal to the convexity of the liabilities. For the position to be fully immunized, the convexity of the assets must be greater than the convexity of the liabilities.

15.05 Questions

Question 15.01

A company has a liability that requires it to make a payment of 600 in one year and 1,500 in two years. The only investments available to the company are the following zero-coupon bonds:

Maturity	Annual Effective Yield	Par Value
1 Year	3%	100
2 Years	4%	100

Calculate the cost to the company to exactly match its liability payments.

A 1,964 B 1,969 C 1,991 D 2,008 E 2,025

Question 15.02

A company has an obligation to pay 5,000 at the end of 1 year and 7,000 at the end of 2 years. The company exactly matches is obligation by purchasing a combination of the following two annual coupon bonds for a total cost of X:

Maturity	Coupon Rate	Annual Effective Yield
1 Year	5%	4%
2 Years	3%	4%

Calculate X.

A 10,742 B 11,266 C 11,280 D 11,360 E 11,558

Question 15.03

A company must pay a liability of 3,000 in one year and 1,200 in two years. The company exactly matches its liabilities with the following investments:

- Loan A: A one-year loan in which X is lent. The loan is repaid with a single payment at time 1. The annual effective interest rate of the loan is 5%.
- Loan B: A two-year loan in which Y is lent. The loan is repaid with two equal annual payments. The annual effective interest rate of the loan is 6%.

Calculate the sum $X+Y$.

A 3,850 B 3,914 C 3,925 D 3,929 E 3,978

Question 15.04

A company must pay a liability of 10,000 in one year and a liability of 15,000 in two years. The following assets are available:

- One-year bond with an annual coupon of 7%, priced at par.
- Two-year bond with an annual coupon of 4%, priced at par.
- Two-year zero-coupon bond with an annual effective yield of 5.5%.

Calculate the minimum cost to the company today to purchase assets that exactly match the liabilities.

A 22,447 B 22,823 C 22,955 D 23,214 E 23,484

Question 15.05

A company must pay liabilities of 1,000 in 6 months and another 1,000 in one year. There are two assets available:

- Bond A: A 6-month bond with a face amount of 1,000, a coupon rate of 5% convertible semiannually, and a 4% yield convertible semiannually.
- Bond B: A 1-year bond with a face amount of 1,000, a coupon rate of 6% convertible semiannually, and a 5% yield convertible semiannually.

Calculate the amount of each bond that the company must purchase to match the liabilities exactly.

A 0.943 of Bond A and 0.971 of Bond B

B 0.947 of Bond A and 0.971 of Bond B

C 0.952 of Bond A and 0.976 of Bond B

D 0.956 of Bond A and 0.976 of Bond B

E 0.976 of Bond A and 0.971 of Bond B

Question 15.06

A company must pay liabilities of 98 in one year, 100 in two years, and 107 in three years. The only investments available are Bond A, Bond B, and Bond C. Bond A and Bond C are annual coupon bonds, and Bond B is a zero-coupon bond. The par value of all three bonds is 100, and all three bonds will be redeemed at par.

Bond	Maturity	Annual Effective Yield	Coupon Rate
A	1 Year	3%	5%
B	2 Years	4%	0%
C	3 Years	5%	4%

Calculate the number of units of Bond A that must be purchased to match the liabilities exactly.

A 0.894 B 0.897 C 0.913 D 0.933 E 0.959

Question 15.07

A bank accepts a deposit of 10,000, and it agrees to pay 4% annual effective interest on the deposit. The depositor will withdraw half of the accumulated balance at the end of 1 year and the remaining accumulated balance at the end of 2 years.

The bank has the following bonds available for purchase:

Bond	Maturity	Annual Effective Yield	Coupon Rate
A	1 Year	4%	0%
B	2 Years	5%	0%
C	2 Years	6%	6%

Any portion of the deposit that is not invested in the bonds is retained by the bank as profit.

The bank develops five strategies for how much to invest in each bond.

Determine which of the following strategies produces the highest immediate profit for the bank and is guaranteed to produce sufficient cash flow to meet the withdrawals of the depositor.

A 4,500 in Bond A, 4,598 in Bond B, and 320 in Bond C

B 5,000 in Bond A and 4,905 in Bond C

C 5,000 in Bond A and 5,000 in Bond B

D 4,706 in Bond A and 5,102 in Bond C

E 4,723 in Bond A, 290 in Bond B, and 4,800 in Bond C

Question 15.08

A company has asset cash flows at time 3 of A and at time 10 of B. It has a liability cash flow of 120,000 at time 6. The company used Redington immunization to immunize the portfolio, based on an annual effective interest rate of 5%

Calculate the ratio $\frac{A}{B}$.

A 0.8041 B 0.9476 C 1.0000 D 1.0553 E 1.2437

Question 15.09

A company has asset cash flows at time 5 of A and at time 10 of B. It has a liability cash flow of L at time 9. The company used Redington immunization to immunize the portfolio, based on an annual effective interest rate of 5%

Calculate the ratio $\frac{B}{A}$.

A 1.5 B 2.0 C 4.0 D 5.0 E 5.1

Question 15.10

Rebecca has liabilities of 374.11 due at the end of each of the next three years. Rebecca will invest 1,000 now to fund these liabilities.

The only investments available are one-year and three-year zero-coupon bonds. The yield curve is flat at an annual effective rate of 6%.

Rebecca matches the duration of her assets to the duration of her liabilities. Determine how much Rebecca invests in each bond.

A 481 in the one-year bond and 519 in the three-year bond

B 490 in the one-year bond and 510 in the three-year bond

C 500 in the one-year bond and 500 in the three-year bond

D 510 in the one-year bond and 490 in the three-year bond

E 519 in the one-year bond and 481 in the three-year bond

Question 15.11

The present value of a company's liabilities is 5,000. The Macaulay duration and Macaulay convexity of the company's liabilities are 12 and 195.

The company creates an investment portfolio consisting of investments in two of the following zero-coupon bonds:

- 5-year bond
- 10-year bond
- 20-year bond

The position satisfies the requirements for Redington immunization. The annual effective yield on each of the bonds is 8%.

Determine which of the following portfolios is created by the company.

A Invest 1,000 in the 5-year bond and 4,000 in the 10-year bond

B Invest 2,000 in the 5-year bond and 3,000 in the 20-year bond

C Invest 2,666.67 in the 5-year bond and 2,333.33 in the 20-year bond

D Invest 3,000 in the 10-year bond and 2,000 in the 20-year bond

E Invest 4,000 in the 10-year bond and 1,000 in the 20-year bond

Question 15.12

As of 12/31/2015, a company has an obligation to pay 1,000,000 on 12/31/2020.

The company purchases a 5-year bond that pays annual coupons of 4%. The purchase price of the bond is equal to its par amount of 821,927.11. The redemption value of the bond is equal to its par value.

On 12/31/2015, the yield curve is level at an annual effective rate of 4%. The yield curve determines the yield available for reinvesting the coupons.

The company considers two scenarios that could occur on 1/1/2016:

- Scenario A: The yield falls to 3.70%.
- Scenario B: The yield rises to 4.30%.

Which of the following best describes the company's profit or loss as of 12/31/2020 after the liability is paid.

A Scenario A: 0 profit/loss. Scenario B: 0 profit/loss.

B Scenario A: 1,434 loss. Scenario B: 1,451 profit.

C Scenario A: 1,064 loss. Scenario B: 1,071 profit.

D Scenario A: 1,071 profit. Scenario B: 1,064 loss.

E Scenario A: 14,340 loss. Scenario B: 14,507 profit.

Question 15.13

A company has a liability of 595.51 due at the end of 3 years and another liability of 709.26 due at the end of 6 years.

An asset portfolio consists of two zero-coupon bonds, Bond A and Bond B.

The annual effective interest rate is 6%.

Which of the asset portfolios below satisfies the conditions of Redington immunization?

A Bond A: 1-year bond with a price of 500
Bond B: 8-year bond with a price of 500

B Bond A: 1-year bond with a price of 400
Bond B: 8-year bond with a price of 600

C Bond A: 1-year bond with a price of 600
Bond B: 8-year bond with a price of 600

D Bond A: 1-year bond with a price of 400
Bond B: 8-year bond with a price of 400

E Bond A: 4.5-year bond with a price of 1,000
Bond B: Not used

Question 15.14

A company has liability cash flows at time 5 of *A* and at time 10 of *B*.

The company has an asset that produces 15 annual payments of 700, with the first payment to be made at time 1.

The annual effective interest rate used to value both the asset and the liabilities is 6.2%. The position satisfies the conditions of Redington immunization.

Calculate *B*.

A 2,540 B 3,356 C 4,172 D 4,634 E 5,635

Question 15.15

A company has liability cash flows of 600 in 1 year and 1,500 in 5 years.

The company sets up its asset portfolio so that the present value and duration of its liabilities is matched by the present value and duration of its assets.

The annual effective interest rate is 5%.

The investment portfolio produces cash flows of *X* now and *Y* in 6 years.

Calculate *X* and determine whether the asset portfolio satisfies the conditions for Redington immunization.

A X = 672 and the conditions for Redington immunization are satisfied.

B X = 672 and the conditions for Redington immunization are not satisfied.

C X = 802 and the conditions for Redington immunization are not satisfied.

D X = 1,075 and the conditions for Redington immunization are satisfied.

E X = 1,075 and the conditions for Redington immunization are not satisfied.

Question 15.16

Determine which of the following statements is false.

A Modified duration is the Macaulay duration divided by the following sum: one plus the annual effective yield.

B When the yield is expressed as a force of interest, the modified duration is equal to the Macaulay duration.

C A cash flow matched portfolio satisfies the first two conditions of Redington immunization.

D A cash flow matched portfolio satisfies the first two conditions of full immunization.

E A fully immunized portfolio satisfies the conditions of Redington immunization.

Question 15.17

A company must pay off a liability of 7,000 in 5 years.

The company has an asset portfolio that produces a payment of A in 2 years and a payment of B in 8 years.

The company employs a full immunization strategy, and the annual effective interest rate is 3.5%.

Calculate $|A-B|$.

A 0 B 270 C 609 D 724 E 860

Question 15.18

A company has a liability of 1,000 that is due 7 years from today.

The annual effective interest rate is 3%. The company purchases 2 assets, and the conditions for full immunization are satisfied.

Asset A will provide a cash flow of 200 in 5 years. Asset B will provide a cash flow of B in y years.

Calculate B.

A 641 B 743 C 800 D 813 E 823

Question 15.19

A company has a liability of 15,000 that is due 9 years from today.

The annual effective interest rate is 4%. The company purchases 2 assets, and the conditions for full immunization are satisfied.

The first asset will provide a cash flow of 7,000 in 4 years. The second asset will provide a cash flow of Y in $(9 + y)$ years.

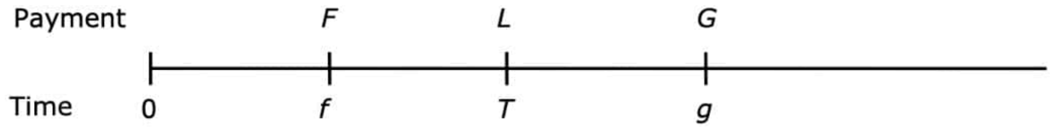
Calculate the ratio $\frac{Y}{y}$.

A 897 B 911 C 1,277 D 1,678 E 3,358

15.06 Appendix – Full Immunization Proof

This proof shows that if the conditions for full immunization are satisfied at the outset, then the surplus becomes positive if the interest rate changes, regardless of the size of the interest rate change.

Assume that two asset cash flows F and G occur at times f and g respectively, and a liability cash flow of L at time T is bracketed by the two asset cash flows, such that $f < T < g$:



As shown above, the cash flows satisfy the third condition for full immunization.

We use r to denote the continuously compounded interest rate and r_0 denotes the current rate. Assume that the cash flows satisfy all of the conditions for full immunization, including the first two conditions shown below:

1. $S(r_0) = 0$
2. $S'(r_0) = 0$

To show that the portfolio is fully immunized, we need to show that:

$$S(r) \geq 0 \quad \text{for any value of } r$$

Showing the following will be sufficient to prove that the cash flows are fully immunized:

$$S(r) > 0 \quad \text{for } r \neq r_0$$

Proof

The value of the surplus and the first derivative of the surplus with respect to r are shown below:

$$S(r) = Fe^{-rf} - Le^{-rT} + Ge^{-rg}$$
$$S'(r) = -fFe^{-rf} + TLe^{-rT} - gGe^{-rg}$$

We are given that the current value of the surplus is 0, which allows us to find an expression for L:

$$S(r_0) = 0$$
$$Fe^{-r_0 f} - Le^{-r_0 T} + Ge^{-r_0 g} = 0$$
$$Fe^{r_0(T-f)} - L + Ge^{r_0(T-g)} = 0$$
$$L = Fe^{r_0(T-f)} + Ge^{r_0(T-g)}$$

The first derivative of the surplus is also equal to zero. In the third line below, the two expression in brackets are each equal to zero, allowing us to add them to the left side of the equation without changing the right side:

$$S'(r_0) = 0$$
$$-fFe^{-r_0 f} + TLe^{-r_0 T} - gGe^{-r_0 g} = 0$$
$$\left[TFe^{-r_0 f} - TFe^{-r_0 f}\right] + \left[TGe^{-r_0 g} - TGe^{-r_0 g}\right] - fFe^{-r_0 f} + TLe^{-r_0 T} - gGe^{-r_0 g} = 0$$
$$T\left[-Fe^{-r_0 f} + Le^{-r_0 T} - Ge^{-r_0 g}\right] + TFe^{-r_0 f} - fFe^{-r_0 f} + TGe^{-r_0 g} - gGe^{-r_0 g} = 0$$
$$-T\left[Fe^{-r_0 f} - Le^{-r_0 T} + Ge^{-r_0 g}\right] + TFe^{-r_0 f} - fFe^{-r_0 f} + TGe^{-r_0 g} - gGe^{-r_0 g} = 0$$

By the first condition for full immunization, the expression in brackets in the final line above is equal to zero. We can now find an expression for G:

$$-T\left[0\right] + TFe^{-r_0 f} - fFe^{-r_0 f} + TGe^{-r_0 g} - gGe^{-r_0 g} = 0$$

$$(T-f)Fe^{-r_0 f} + (T-g)Ge^{-r_0 g} = 0$$

$$(T-g)Ge^{-r_0 g} = -(T-f)Fe^{-r_0 f}$$

$$G = -\frac{(T-f)Fe^{-r_0 f}}{(T-g)e^{-r_0 g}}$$

$$G = \frac{(T-f)Fe^{r_0(g-f)}}{(g-T)}$$

Using the expression for L found above, we can substitute for L in the expression for the value of the surplus:

$$\begin{aligned}
S(r) &= Fe^{-rf} - Le^{-rT} + Ge^{-rg} \\
&= Fe^{-rf} - \left(Fe^{r_0(T-f)} + Ge^{r_0(T-g)}\right)e^{-rT} + Ge^{-rg} \\
&= Fe^{-rf} - Fe^{r_0(T-f)} - Ge^{r_0(T-g)} + Ge^{-rg} \\
&= e^{-rT}\left[Fe^{r(T-f)} - Fe^{r_0(T-f)} - Ge^{r_0(T-g)} + Ge^{r(T-g)}\right] \\
&= e^{-rT}\left[Fe^{r(T-f)} - Fe^{r_0(T-f)} + G\left(e^{r(T-g)} - e^{r_0(T-g)}\right)\right]
\end{aligned}$$

Using the expression for G found above, we can substitute for G:

$$\begin{aligned}
S(r) &= e^{-rT}\left[Fe^{r(T-f)} - Fe^{r_0(T-f)} + G\left(e^{r(T-g)} - e^{r_0(T-g)}\right)\right] \\
&= e^{-rT}\left[Fe^{r(T-f)} - Fe^{r_0(T-f)} + \frac{(T-f)Fe^{r_0(g-f)}}{(g-T)}\left(e^{r(T-g)} - e^{r_0(T-g)}\right)\right] \\
&= e^{-rT}Fe^{r_0(T-f)}\left[e^{r(T-f)}e^{-r_0(T-f)} - 1 + \frac{(T-f)e^{r_0(g-f)}e^{-r_0(T-f)}}{(g-T)}\left(e^{r(T-g)} - e^{r_0(T-g)}\right)\right] \\
&= e^{-rT}Fe^{r_0(T-f)}\left[e^{(r-r_0)(T-f)} - 1 + \frac{(T-f)e^{r_0(g-T)}}{(g-T)}\left(e^{r(T-g)} - e^{r_0(T-g)}\right)\right] \\
&= e^{-rT}Fe^{r_0(T-f)}\left[e^{(r-r_0)(T-f)} - 1 + \frac{(T-f)}{(g-T)}\left(e^{-(r-r_0)(g-T)} - 1\right)\right]
\end{aligned}$$

Define the portion in brackets as:

$$f(r) = e^{(r-r_0)(T-f)} - 1 + \frac{(T-f)}{(g-T)}\left(e^{-(r-r_0)(g-T)} - 1\right)$$

We have:

$$S(r) = e^{-rT}Fe^{r_0(T-f)} \times f(r)$$

The first portion of the expression above, $e^{-rT}Fe^{r_0(T-f)}$, is positive for all values of r. Therefore, to show that $S(r) > 0$ when $r \neq r_0$, we need only show that $f(r) > 0$ when $r \neq r_0$.

We begin by observing that $f(r)$ approaches infinity as r becomes very large:

$$\lim_{r\to\infty}\left[f(r)\right] = \lim_{r\to\infty}\left[e^{(r-r_0)(T-f)} - 1 + \frac{(T-f)}{(g-T)}\left(e^{-(r-r_0)(g-T)} - 1\right)\right]$$

$$= \lim_{r\to\infty}\left[e^{r(T-f)}e^{-r_0(T-f)} - 1 + \frac{(T-f)}{(g-T)}(1-1)\right] = \lim_{r\to\infty}\left[e^{r(T-f)}e^{-r_0(T-f)} - 1\right] = \infty$$

The function also approaches infinity as r becomes very small:

$$\lim_{r\to-\infty}\left[f(r)\right] = \lim_{r\to-\infty}\left[e^{(r-r_0)(T-f)} - 1 + \frac{(T-f)}{(g-T)}\left(e^{-(r-r_0)(g-T)} - 1\right)\right]$$

$$= \lim_{r\to-\infty}\left[1 - 1 + \frac{(T-f)}{(g-T)}\left(e^{-r(g-T)}e^{r_0(g-T)} - 1\right)\right]$$

$$= \lim_{r\to-\infty}\left[\frac{(T-f)}{(g-T)}\left(e^{-r(g-T)}e^{r_0(g-T)} - 1\right)\right] = \infty$$

Next we find the critical numbers of $f(r)$:

$$f'(r) = (T-f)\left[e^{(r-r_0)(T-f)} - e^{-(r-r_0)(g-T)}\right]$$

$$0 = (T-f)\left[e^{(r-r_0)(T-f)} - e^{-(r-r_0)(g-T)}\right]$$

$$0 = e^{(r-r_0)(T-f)} - e^{-(r-r_0)(g-T)}$$

$$e^{-(r-r_0)(g-T)} = e^{(r-r_0)(T-f)}$$

$$-(r-r_0)(g-T) = (r-r_0)(T-f)$$

Since $(g-T)$ and $(T-f)$ are both positive, the two sides of the equation above have opposite signs unless both sides are zero, and that can only occur if:

$$r = r_0$$

Therefore, there is just one critical number, and it is $r = r_0$.

When $r = r_0$, we have:

$$f(r_0) = e^{(r_0-r_0)(T-f)} - 1 + \frac{(T-f)}{(g-T)}\left(e^{-(r_0-r_0)(g-T)} - 1\right) = 1 - 1 + \frac{(T-f)}{(g-T)}(1-1) = 0$$

The second derivative of $f(r)$ is found below:

$$f'(r) = (T-f)\left[e^{(r-r_0)(T-f)} - e^{-(r-r_0)(g-T)}\right]$$

$$f''(r) = (T-f)\left[(T-f)e^{(r-r_0)(T-f)} + (g-T)e^{-(r-r_0)(g-T)}\right]$$

Since the second derivative is positive for all values of r, the function is convex. Therefore, $f(r_0) = 0$ is the minimum value of $f(r)$, and:

$$f(r) > 0 \quad \text{for } r \neq r_0$$

Recall that $S(r)$ is a positive value times $f(r)$:

$$S(r) = e^{-rT}Fe^{r_0(T-f)} \times f(r)$$

Therefore, we conclude that the minimum of $S(r)$ occurs at $r = r_0$ and:

$$S(r) > 0 \quad \text{for } r \neq r_0$$

This proves that if the conditions of full immunization are satisfied at the outset, then the surplus becomes positive if the interest rate changes, regardless of the size of the interest rate change.

Although we have assumed that there are just two asset cash flows, the proof is easily extended to the case of more than two asset cash flows. This is accomplished by breaking the portfolio down into portfolio segments, each of which has just two asset cash flows and one liability cash flow.

Chapter 16: Term Structure of Interest Rates

Interest rates can vary according to the interval of time, or term, to which they apply. The **term structure of interest rates** can be described by the following:

- yield curve
- spot rate curve
- forward rates

Although interest rates can change over time, this chapter's examination of the term structure of interest rates is an examination of only the interest rates that are known at the current time.

16.01 Yields

The yield of a bond depends on its time to maturity.

Example 16.01

The table below provides the times until maturity, prices, and coupon rates of 10 bonds. Each of the bonds makes annual coupon payments and has a face amount of $100.

Maturity	Bond Price	Annual Coupon
1	105.8824	8.000%
2	104.8186	5.000%
3	97.1714	2.000%
4	85.4804	0.000%
5	106.5850	6.000%
6	110.1514	7.000%
7	77.6705	2.000%
8	104.6116	7.000%
9	98.3537	6.500%
10	114.0472	9.000%

Calculate the annual effective yield of each bond.

Solution

The yield of each bond is the internal rate of return that equates the bond's present values to its price:

$$P = Coup \times a_{\overline{n}|y} + Rv^n$$

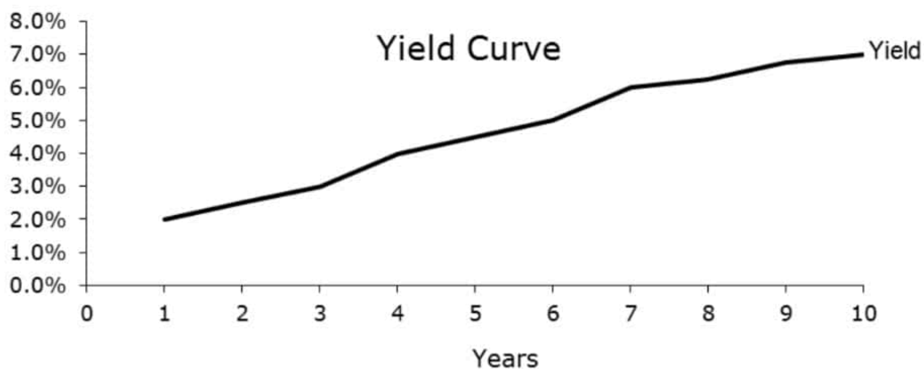
We can use the BA II Plus calculator to find the yields.

1 [*N*] −105.8824 [*PV*] 8 [*PMT*] 100 [*FV*] [CPT] [*I/Y*] Yield is **2.00%**.
2 [*N*] −104.8186 [*PV*] 5 [*PMT*] [CPT] [*I/Y*] Yield is **2.50%**.
3 [*N*] −97.1714 [*PV*] 2 [*PMT*] [CPT] [*I/Y*] Yield is **3.00%**.
4 [*N*] −85.4804 [*PV*] 0 [*PMT*] [CPT] [*I/Y*] Yield is **4.00%**.
5 [*N*] −106.5850 [*PV*] 6 [*PMT*] [CPT] [*I/Y*] Yield is **4.50%**.
6 [*N*] −110.1514 [*PV*] 7 [*PMT*] [CPT] [*I/Y*] Yield is **5.00%**.
7 [*N*] −77.6705 [*PV*] 2 [*PMT*] [CPT] [*I/Y*] Yield is **6.00%**.
8 [*N*] −104.6116 [*PV*] 7 [*PMT*] [CPT] [*I/Y*] Yield is **6.25%**.
9 [*N*] −98.3537 [*PV*] 6.5 [*PMT*] [CPT] [*I/Y*] Yield is **6.75%**.
10 [*N*] −114.0472 [*PV*] 9 [*PMT*] [CPT] [*I/Y*] Yield is **7.00%**.

The table in the example above is repeated below with the yields in the rightmost column.

Maturity	Bond Price	Annual Coupon	Yield
1	105.8824	8.000%	2.000%
2	104.8186	5.000%	2.500%
3	97.1714	2.000%	3.000%
4	85.4804	0.000%	4.000%
5	106.5850	6.000%	4.500%
6	110.1514	7.000%	5.000%
7	77.6705	2.000%	6.000%
8	104.6116	7.000%	6.250%
9	98.3537	6.500%	6.750%
10	114.0472	9.000%	7.000%

When the yields are graphed against their times until maturity, the resulting graph is called a yield curve:



A yield applies to a particular set of cash flows. It is the internal rate of return for that set of cash flows.

16.01

Yield

The present value of a stream of cash flows is the sum of the discounted values of those cash flows, where the cash flows are discounted at the yield corresponding to the pattern of cash flows:

$$PV = \sum_{t>0} \frac{CF_t}{(1+y)^t}$$

Example 16.02 A bond makes annual coupon payments of \$6 per year for 5 years. At the end of 5 years, it matures for its redemption value of \$100. The annual effective yield on the bond is 4.5%.

Calculate the price of the bond.

Solution The price of the bond is:

$$PV = \sum_{t>0} \frac{CF_t}{(1+y)^t} = Coup \times a_{\overline{n}|y} + Rv^n = 6a_{\overline{5}|0.045} + \frac{100}{1.045^5}$$

$$= 6 \times \frac{1-1.045^{-5}}{0.045} + 80.2451 = 6 \times 4.3900 + 80.2451 = \mathbf{106.5850}$$

Alternatively, we can use the BA II Plus calculator to find the price.

5 [*N*] 4.5 [*I/Y*] 6 [*PMT*] 100 [*FV*] [CPT] [*PV*]

Result is −106.5850. Price is **106.5850**.

Not all streams of cash flows have the same yield, but a multiple of a stream of cash flows has the same yield as the original stream of cash flows. Another way to say this is that all cash flow streams that follow the same cash flow pattern have the same yield.

Example 16.03 A bond makes annual coupon payments of $6 per year for 5 years. At the end of 5 years, it matures for its redemption value of $100. The price of the bond is $106.5850.

a. Calculate the annual effective yield for an investor that owns 1 of the bonds.

b. Calculate the annual effective yield for an investor that owns 2 of the bonds.

Solution We can use the BA II Plus calculator to find the yield of the positions.

a. Below, we find the yield for one bond:

5 [*N*] −106.5850 [*PV*] 6 [*PMT*] 100 [*FV*] [CPT] [*I*/*Y*]

Yield is **4.50%**.

b. Below, we find the yield for two bonds. As expected, the yield is equal to yield of one bond:

5 [*N*] −106.5850 × 2 [*PV*] 12 [*PMT*] 200 [*FV*] [CPT] [*I*/*Y*]

Yield is **4.50%**.

A **par bond yield curve** is convenient because the its yields are equal to the coupon rates of the bonds. That is, the coupon rate of each bond is equal to the bond's yield. Since most bonds are not priced at par in the marketplace, however, par bond yield curves cannot be observed and must be estimated.

Example 16.04 The table below provides the times until maturity, prices, and coupon rates of 10 bonds. Each of the bonds makes annual coupon payments and has a face amount of $100.

Maturity	Bond Price	Annual Coupon
1	100.0000	2.0000%
2	100.0000	2.4938%
3	100.0000	2.9803%
4	100.0000	3.9219%
5	100.0000	4.3855%
6	100.0000	4.8332%
7	100.0000	5.6618%
8	100.0000	5.8837%
9	100.0000	6.2719%
10	100.0000	6.4730%

Calculate the annual effective yield of each bond.

Solution Since the price of each bond is equal to its par value, the yield of each bond must be equal to its coupon rate. The table below contains the yields in its rightmost column.

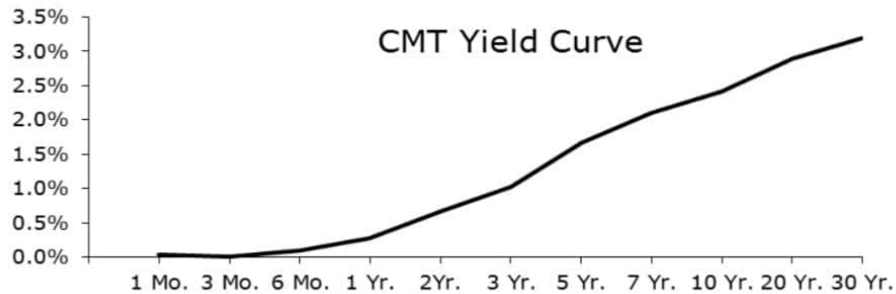
Maturity	Bond Price	Annual Coupon	Yield
1	100.0000	2.0000%	2.0000%
2	100.0000	2.4938%	2.4938%
3	100.0000	2.9803%	2.9803%
4	100.0000	3.9219%	3.9219%
5	100.0000	4.3855%	4.3855%
6	100.0000	4.8332%	4.8332%
7	100.0000	5.6618%	5.6618%
8	100.0000	5.8837%	5.8837%
9	100.0000	6.2719%	6.2719%
10	100.0000	6.4730%	6.4730%

If we were to graph the yields against the time until maturity, the resulting yield curve would be a par bond yield curve.

Constant Maturity Treasury Series

The Constant Maturity Treasury (CMT) yield curve is published daily by the United States Treasury Department, and it is based on the yields of U.S. Treasuries with times to maturity from 1 month to 30 years. The yields are based on the prices observed for the most recently issued Treasuries, which are also known as **on-the-run** Treasuries. On-the-run Treasuries typically trade close to par, so the CMT yield curve can be considered a par yield curve.

The CMT yield curve as of 7/14/2015 is shown below.



16.02 Spot Rates

A **spot rate** is equal to the yield on a zero-coupon bond, and it is used to discount a single cash flow. Unless we are told otherwise, we assume that spot rates are expressed as annual effective rates of interest.

Investment firms can create zero-coupon bonds by stripping off and selling the individual coupon payments of a U.S. Treasury bond. The resulting zero-coupon bonds are known as Treasury strips or strip bonds.

16.02

Spot Rates

The present value of a stream of cash flows is the sum of the discounted values of those cash flows, where the cash flows are discounted at the spot rates corresponding to the cash flows:

$$PV = \sum_{t>0} \frac{CF_t}{(1+s_t)^t}$$

where:

s_t = spot rate for a cash flow occuring at time t

Example 16.05 The annual effective 1-year spot rate is 2%, and the annual effective 2-year spot rate is 2.512%. Find the value of a 2-year bond that pays annual coupons at a rate of 5% and has a par value of $100.

Solution The price of the bond is found by discounting the 1-year cash flow at the 1-year spot rate and the 2-year cash flow at the 2-year spot rate:

$$PV = \sum_{t>0} \frac{CF_t}{(1+s_t)^t} = \frac{5}{1.02} + \frac{105}{1.02512^2} = \mathbf{104.82}$$

Yields can be converted into spot rates through a process known as **bootstrapping**, which makes use of the fact that both the yields and the spot rates must produce the same present values:

$$\sum_{t>0} \frac{CF_t}{(1+y)^t} = \sum_{t>0} \frac{CF_t}{(1+s_t)^t}$$

We begin with the shortest bond, and work our way to the longest bond, one bond at a time. The shortest bond has just one cash flow, so its yield is also a spot rate. We can use this spot rate along with the yield of the next shortest bond to find the next spot rate. This continues until all of the spot rates have been determined.

Example 16.06

The table below provides the times until maturity, coupon rates, and annual effective yields. Each of the bonds makes annual coupon payments and has a face amount of $100.

Maturity	Annual Coupon	Yield
1	8.000%	2.000%
2	5.000%	2.500%
3	2.000%	3.000%
4	0.000%	4.000%
5	6.000%	4.500%
6	7.000%	5.000%
7	2.000%	6.000%
8	7.000%	6.250%
9	6.500%	6.750%
10	9.000%	7.000%

Calculate the 1-year, 2-year, 3-year, 4-year, and 5-year annual effective spot rates.

Solution

The 1-year bond has only one cash flow, so its yield is equal to the 1-year spot rate:

$$\frac{108}{1.02} = \frac{108}{1+s_1} \quad \Rightarrow \quad s_1 = \mathbf{0.02}$$

The 2-year spot rate is found below:

$$\frac{5}{1.025} + \frac{105}{1.025^2} = \frac{5}{1.02} + \frac{105}{(1+s_2)^2} \quad \Rightarrow \quad s_2 = \mathbf{0.02512}$$

The 3-year spot rate is found below:

$$\frac{2}{1.03} + \frac{2}{1.03^2} + \frac{102}{1.03^3} = \frac{2}{1.02} + \frac{2}{1.02512^2} + \frac{102}{(1+s_3)^3} \quad \Rightarrow \quad s_3 = \mathbf{0.03014}$$

The 4-year spot rate is easy to find because the 4-year bond is zero-coupon bond:

$$\frac{100}{1.04^4} = \frac{100}{(1+s_4)^4} \quad \Rightarrow \quad s_4 = \mathbf{0.04000}$$

The 5-year spot rate is found below:

$$6a_{\overline{5}|0.045} + \frac{100}{1.045^5} = \frac{6}{1.02} + \frac{6}{1.02512^2} + \frac{6}{1.03014^3} + \frac{6}{1.04^4} + \frac{106}{(1+s_5)^5}$$

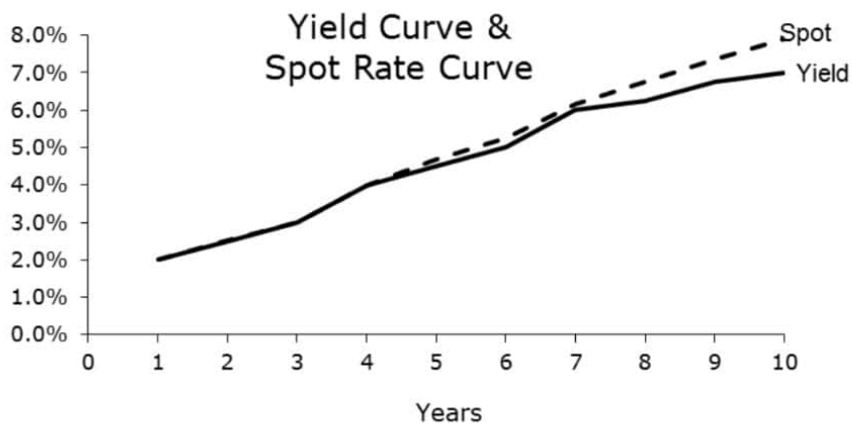
$$106.5850 = \frac{6}{1.02} + \frac{6}{1.02512^2} + \frac{6}{1.03014^3} + \frac{6}{1.04^4} + \frac{106}{(1+s_5)^5}$$

$$\Rightarrow \quad s_5 = \mathbf{0.04669}$$

The spot rates calculated above and the remaining 5 spot rates are shown in the rightmost column in the table below.

Maturity	Annual Coupon	Yield	Spot
1	8.000%	2.000%	2.000%
2	5.000%	2.500%	2.512%
3	2.000%	3.000%	3.014%
4	0.000%	4.000%	4.000%
5	6.000%	4.500%	4.669%
6	7.000%	5.000%	5.260%
7	2.000%	6.000%	6.142%
8	7.000%	6.250%	6.767%
9	6.500%	6.750%	7.359%
10	9.000%	7.000%	7.903%

The graph below shows the yield curve and the spot rate curve.



We usually write the present value of an annuity in terms of a single yield, but we can also write it in terms of the spot rates:

$$a_{\overline{n}|spots} = \frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + \cdots + \frac{1}{(1+s_n)^n}$$

A discount factor that is calculated using a spot rate is denoted by v^n_{spot}. This is equal to the price of a zero-coupon bond that matures at time n for 1, which is denoted by P_n.

$$P_n = v^n_{spot} = \left(\frac{1}{1+s_n}\right)^n$$

A yield applies to a specific stream of cash flows, while a spot rate applies to a cash flow that occurs at the specified time. Therefore, we have to be careful when valuing a stream of cash flows to use either the yield for those cash flows or the spot rates, not a mix of both.

The expression below shows that yields or spots can be used to find the price of a bond:

$$Coup \times a_{\overline{n}|spots} + \frac{R}{(1+s_n)^n} = Coup \times a_{\overline{n}|y} + \frac{R}{(1+y)^n}$$

But in general, unless the yield curve is flat, we have:

$$a_{\overline{n}|spots} \neq a_{\overline{n}|y} \quad \text{and} \quad \frac{1}{(1+s_n)^n} \neq \frac{1}{(1+y)^n}$$

When using $a_{\overline{n|}spots}$ and v_{spot}^{n}, it usually isn't difficult to remember that they are based on spot rates instead of yields, so they are often written as $a_{\overline{n|}}$ and v^{n}.

The example below shows that if we have the spot rate for a $(n-1)$-year bond, then we can use it to find the spot rate of an n–year bond.

Example 16.07

The table below provides the times until maturity, coupon rates, and annual effective yields for 2 bonds. Each of the bonds makes annual coupon payments and has a face amount of $100.

Maturity	Annual Coupon	Yield
5	6.000%	4.500%
6	7.000%	5.000%

You are given that the annual effective 5-year spot rate is 4.669%. Calculate the annual effective 6-year spot rate.

Solution

We can use the BA II Plus calculator to find the prices of the 5-year bond and the 6-year bond:

5 [*N*] 4.5 [*I/Y*] 6 [*PMT*] 100 [*FV*] [CPT] [*PV*]

Result is –106.5850. Price of 5-year bond is 106.5850.

6 [*N*] 5 [*I/Y*] 7 [*PMT*] 100 [*FV*] [CPT] [*PV*]

Result is –110.1514. Price of 6-year bond is 110.1514.

We can write the price of the 5-year bond in terms of the spot rates, and therefore we can find the present value of a 5-year annuity-immediate:

$$6a_{\overline{5|}spots} + 100v_{spot}^{5} = 106.5850$$

$$6a_{\overline{5|}spots} + \frac{100}{1.04669^{5}} = 106.5850$$

$$a_{\overline{5|}spots} = 4.4976$$

We can also write the price of the 6-year bond in terms of the spot rates. Below we use a 5-year annuity to value the first 5 coupons, and we include the final coupon with the redemption amount:

$$7a_{\overline{5|}spots} + 107v_{spot}^{6} = 110.1514$$

$$7 \times 4.4976 + 107v_{spot}^{6} = 110.1514$$

$$v_{spot}^{6} = 0.7352$$

$$\frac{1}{(1+s_6)^6} = 0.7352$$

$$s_6 = \mathbf{0.05260}$$

We can use the spot rates to find the coupon rate of an n-year bond that is priced at par:

$$F = cF \times a_{\overline{n|}spots} + Fv_{spot}^{n}$$

$$1 = c \times a_{\overline{n|}spots} + v_{spot}^{n}$$

$$c = \frac{1 - v_{spot}^{n}}{a_{\overline{n|}spots}}$$

16.03

Yield on a Par Bond

The coupon rate and yield of an n-year bond that is priced at par is:

$$c = \frac{1 - v_{spot}^n}{a_{\overline{n}|spots}} = \frac{1 - P_n}{P_1 + P_2 + \cdots + P_n}$$

Example 16.08

A 10-year bond pays annual coupons of 9% per year. It has a face amount of \$100, and it is priced to have an annual effective yield of 7%. The annual effective 10-year spot rate is 7.903%.

Calculate the coupon rate for a 10-year bond priced at par.

Solution

We can use the BA II Plus calculator to find the price of the 10-year bond:

10 [*N*] 7 [*I/Y*] 9 [*PMT*] 100 [*FV*] [CPT] [*PV*]

Result is −114.0472. Price of 10-year bond is 114.0472.

We can write the price of the 10-year bond in terms of the spot rates, and therefore we can find the present value of a 10-year annuity-immediate:

$$9a_{\overline{10}|spots} + 100v_{spot}^{10} = 114.0472$$

$$9a_{\overline{10}|spots} + \frac{100}{1.07903^{10}} = 114.0472$$

$$a_{\overline{10}|spots} = 7.4789$$

The coupon of a 10-year bond that is priced at par is:

$$c = \frac{1 - v_{spot}^n}{a_{\overline{n}|spot}} = \frac{1 - 1.07903^{-10}}{7.4789} = \mathbf{0.07122}$$

16.03 Forward Rates

A **forward rate** is an interest rate that can be locked in now to apply to a future time interval. When the forward rate applies from time t to time $(t + k)$, the forward rate is denoted as follows:

$$f_{t,t+k} = \text{forward rate applicable from time } t \text{ to time } (t + k)$$

When the unit of time is one year, this rate can also be described as follows:

- A k–year spot rate that comes into effect t years in the future
- k-year forward rate, deferred for t years
- k-year forward rate, starting in t years

Some people refer $f_{t,t+k}$ to as the t-year forward rate. That contradicts the convention of this text, but you might see it elsewhere.

Unless we are told otherwise, we assume that forward rates are expressed as annual effective rates of interest.

When k is 1, we drop the second subscript:

$$f_t = f_{t,t+1} = \text{forward rate applicable from time } t \text{ to time } (t + 1)$$

This rate is sometimes referred to as the one-year forward rate for the (t+1)th year.

16.04

Forward Rates

The present value of a stream of n cash flows is the sum of the discounted values of those cash flows, where the cash flows are discounted at the forward rates:

$$PV_0 = \frac{CF_1}{1+f_0} + \frac{CF_2}{(1+f_0)(1+f_1)} + \cdots + \frac{CF_n}{(1+f_0)(1+f_1)\cdots(1+f_{n-1})}$$

The initial forward rate applies from time 0 to time 1, so it is equal to the time-1 spot rate:

$$f_0 = s_1$$

Example 16.09 The annual effective forward rates are:

$$f_0 = 2\% \qquad f_1 = 3.027\% \qquad f_2 = 4.024\%$$

Find the value of a 3-year bond that pays annual coupons at a rate of 2% and has a par value of $100.

Solution The price of the bond is found by discounting the 1-year cash flow at the initial forward rate. The 2-year cash flow is discounted using the initial forward rate and the forward rate starting in 1 year. The 3-year cash flow is discounted using the initial forward rate, the forward rate starting in 1 year, and the forward rate starting in 2 years:

$$PV = \frac{2}{1.02} + \frac{2}{(1.02)(1.03027)} + \frac{102}{(1.02)(1.03027)(1.04024)} = \mathbf{97.17}$$

Forward rates can be used to accumulate from one time period to another.

Example 16.10 The annual effective forward rates are:

$$f_0 = 2\% \qquad f_1 = 3.027\% \qquad f_2 = 4.024\%$$

In one year a lender will lend $1,000 for 2 years. The loan will be repaid with a single payment of X at the end of 3 years. Calculate the amount of the loan repayment.

Solution The loan is made at time 1 and continues until time 3. The interest rate that applies from time 1 to time 2 is 3.027%, and the interest rate that applies from time 2 to time 3 is 4.024%. Using those two rates to accumulate the value of the loan, we have:

$$X = AV_3 = 1{,}000 \times 1.03027 \times 1.04024 = \mathbf{1{,}071.73}$$

If a borrower issues a bond that locks in the forward rates, then the initial value of the bond is equal to the par value of the bond.

Example 16.11 The annual effective forward rates are:

$$f_0 = 2\% \qquad f_1 = 3.027\% \qquad f_2 = 4.024\%$$

Find the value of a 3-year bond with a par value of $100 that pays annual coupons at a rate of 2% in the first year, 3.027% in the second year, and 4.024% in the third year.

Solution The price of the bond is found by discounting the 1-year cash flow at the initial forward rate. The 2-year cash flow is discounted using the initial forward rate and the forward rate starting in 1 year.

The 3-year cash flow is discounted using the initial forward rate, the forward rate starting in 1 year, and the forward rate starting in 2 years:

$$PV = 100\left[\frac{0.02}{1.02} + \frac{0.03027}{(1.02)(1.03027)} + \frac{\cancel{1.04024}}{(1.02)(1.03027)\cancel{(1.04024)}}\right]$$

$$= 100\left[\frac{0.02}{1.02} + \frac{0.03027}{(1.02)(1.03027)} + \frac{1}{(1.02)(1.03027)}\right]$$

$$= 100\left[\frac{0.02}{1.02} + \frac{\cancel{1.03027}}{(1.02)\cancel{(1.03027)}}\right] = 100\left[\frac{0.02}{1.02} + \frac{1}{(1.02)}\right]$$

$$= 100\left[\frac{\cancel{1.02}}{\cancel{1.02}}\right] = \mathbf{100}$$

Yields can be converted into forward rates by making use of the fact that both the yields and the forward rates must produce the same present values:

$$\sum_{t>0} \frac{CF_t}{(1+y)^t} = \sum_{t>0} \frac{CF_t}{(1+f_0)(1+f_1)\cdots(1+f_{t-1})}$$

We begin with the shortest bond, and work our way to the longest bond, one bond at a time. The shortest bond has just one cash flow, so its yield is also the initial forward rate. We can use this initial forward rate along with the yield of the next shortest bond to find the next forward rate. This continues until all of the forward rates have been determined.

Example 16.12 The table below provides the times until maturity, coupon rates, and annual effective yields. Each of the bonds makes annual coupon payments and has a face amount of $100.

Maturity	Annual Coupon	Yield
1	8.000%	2.000%
2	5.000%	2.500%
3	2.000%	3.000%
4	0.000%	4.000%
5	6.000%	4.500%
6	7.000%	5.000%
7	2.000%	6.000%
8	7.000%	6.250%
9	6.500%	6.750%
10	9.000%	7.000%

Calculate the values of $f_0, f_1, \cdots f_3$.

Solution The 1-year bond has only one cash flow, so its yield is equal to the initial forward rate:

$$\frac{108}{1.02} = \frac{108}{1+f_0} \quad \Rightarrow \quad f_0 = \mathbf{0.02}$$

The 1-year forward rate deferred for 1 year is found below:

$$\frac{5}{1.025} + \frac{105}{1.025^2} = \frac{5}{1.02} + \frac{105}{1.02(1+f_1)} \quad \Rightarrow \quad f_1 = \mathbf{0.03027}$$

The 1-year forward rate deferred for 2 years is found below:

$$\frac{2}{1.03} + \frac{2}{1.03^2} + \frac{102}{1.03^3} = \frac{2}{1.02} + \frac{2}{(1.02)(1.03027)} + \frac{102}{(1.02)(1.03027)(1+f_2)}$$

$$f_2 = \mathbf{0.04024}$$

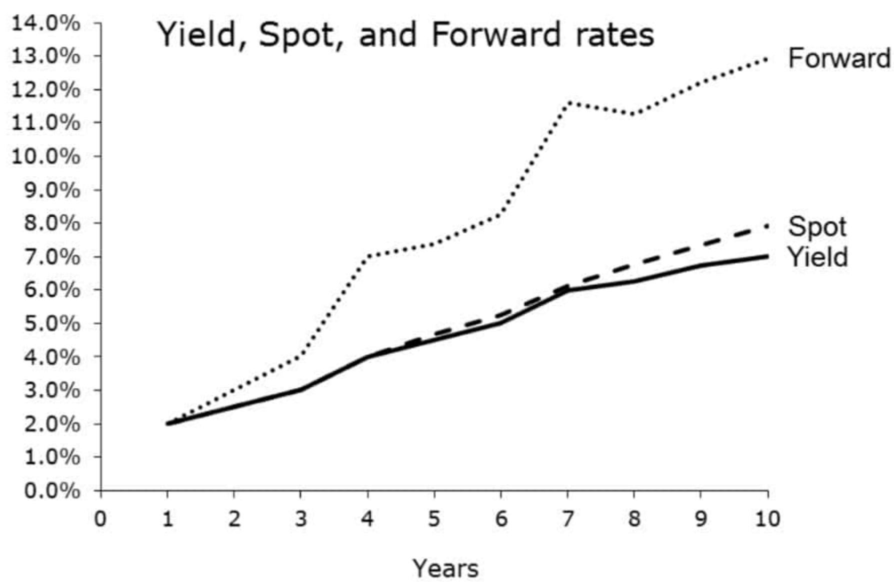
The 1-year forward rate deferred for 3 years is found below:

$$\frac{100}{1.04^4} = \frac{100}{(1.02)(1.03027)(1.04024)(1+f_3)} \quad \Rightarrow \quad f_3 = \mathbf{0.07016}$$

The forward rates calculated above and the remaining 6 forward rates are shown in the rightmost column in the table below.

t	Annual Coupon	Yield	s_t	f_{t-1}
1	8.000%	2.000%	2.000%	2.000%
2	5.000%	2.500%	2.512%	3.027%
3	2.000%	3.000%	3.014%	4.024%
4	0.000%	4.000%	4.000%	7.016%
5	6.000%	4.500%	4.669%	7.388%
6	7.000%	5.000%	5.260%	8.266%
7	2.000%	6.000%	6.142%	11.588%
8	7.000%	6.250%	6.767%	11.250%
9	6.500%	6.750%	7.359%	12.214%
10	9.000%	7.000%	7.903%	12.928%

The graph below shows the yield curve, the spot rate curve, and the forward rate curve.



The present value of a cash flow is the same regardless of whether the present value is calculated using a spot rate or forward rates. As shown below, this equivalence allows us to convert forward rates into a spot rate:

$$\frac{CF_t}{(1+f_0)(1+f_1)\cdots(1+f_{t-1})} = \frac{CF_t}{(1+s_t)^t}$$

$$\frac{1}{(1+f_0)(1+f_1)\cdots(1+f_{t-1})} = \frac{1}{(1+s_t)^t}$$

$$(1+f_0)(1+f_1)\cdots(1+f_{t-1}) = (1+s_t)^t$$

We can also find a forward rate in terms of the spot rates by considering times $(t-1)$ and t:

$$\text{Time } (t-1): \quad (1+f_0)(1+f_1)\cdots(1+f_{t-2}) = (1+s_{t-1})^{t-1}$$

$$\text{Time } t: \quad (1+f_0)(1+f_1)\cdots(1+f_{t-1}) = (1+s_t)^t$$

Dividing the first equation above into the second equation, we have:

$$1+f_{t-1}=\frac{(1+s_t)^t}{(1+s_{t-1})^{t-1}}$$

16.05

Forward Rates and Spot Rates

The following relationships can be used to find spot rates using forward rates and vice versa:

$$(1+s_t)^t=(1+f_0)(1+f_1)\cdots(1+f_{t-1})$$

$$1+f_{t-1}=\frac{(1+s_t)^t}{(1+s_{t-1})^{t-1}}$$

Example 16.13

You are given the following annual effective spot rates:

$s_4=4.000\%$ $\qquad$ $s_5=4.669\%$

Find the 4-year forward rate, deferred for 1 year.

Solution

The 4-year forward rate, deferred for 1 year is the rate that applies from time 4 to time 5:

$$1+f_4=\frac{(1+s_5)^5}{(1+s_4)^4}$$

$$1+f_4=\frac{(1.04669)^5}{(1.04)^4}$$

$$f_4=\mathbf{7.388\%}$$

Although we usually see forward rates that apply for a period of 1 unit of time, it is possible for forward rates to apply for periods of other lengths. Below, we consider a more general form of the formulas in the Key Concepts above.

The present value of a cash flow that occurs at time $(k_1+k_2\cdots+k_n)$ *can be written as:*

$$PV=\frac{CF_{k_1+\cdots+k_n}}{\left(1+f_{0,k_1}\right)^{k_1}\left(1+f_{k_1,k_1+k_2}\right)^{k_2}\cdots\left(1+f_{k_1+k_2+\cdots+k_{n-1},k_1+k_2+\cdots+k_n}\right)^{k_n}}$$

Furthermore, we have:

$$(1+s_{k_1+k_2+\cdots+k_n})^{(k_1+k_2\cdots+k_n)}=\left(1+f_{0,k_1}\right)^{k_1}\left(1+f_{k_1,k_1+k_2}\right)^{k_2}\cdots\left(1+f_{k_1+k_2+\cdots+k_{n-1},k_1+k_2+\cdots+k_n}\right)^{k_n}$$

$$\left(1+f_{k_1+k_2+\cdots+k_{n-1},k_1+k_2+\cdots+k_n}\right)^{k_n}=\frac{(1+s_{k_1+k_2+\cdots+k_n})^{(k_1+k_2+\cdots+k_n)}}{(1+s_{k_1+k_2+\cdots+k_{n-1}})^{(k_1+k_2+\cdots+k_{n-1})}}$$

To use the formulas just above to reproduce the formulas in the Key Concept above, we can make the following substitutions into the formulas just above:

$k_j=1$ for all $j=1,2,\cdots,n$

$n=t$

Don't spend too much time on the formulas just above since they are rarely needed. They are shown to generalize the concept. Forward rates are usually assumed to apply for 1 unit of time, so the formulas in the Key Concepts are adequate for most purposes.

The present value of an annuity expressed in terms of the spot rates is equivalent to the present value expressed in terms of the forward rates:

$$a_{\overline{n}|spots} = \frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + \cdots + \frac{1}{(1+s_n)^n}$$
$$= \frac{1}{1+f_0} + \frac{1}{(1+f_0)(1+f_1)} + \cdots + \frac{1}{(1+f_0)(1+f_1)\cdots(1+f_{n-1})}$$

We can also write v_{spot}^n in terms of the forward rates:

$$v_{spot}^n = \left(\frac{1}{1+s_n}\right)^n = \frac{1}{(1+f_0)(1+f_1)\cdots(1+f_{n-1})}$$

We could define new present value factors for forward rates, and they would be equal to the present value factors based on the spot rates:

$$a_{\overline{n}|fwds} = a_{\overline{n}|spots}$$
$$v_{fwd}^n = v_{spot}^n$$

To avoid unnecessary notation, however, we use $a_{\overline{n}|spots}$ *and* v_{spot}^n *even when working with forward rates.*

Example 16.14

The table below provides times until maturity and coupon rates for two bonds. Both of the bonds makes annual coupon payments and has a face amount of \$100. The annual effective yield of the 5-year bond is also provided.

Maturity	Annual Coupon	Yield
5	6.000%	4.500%
6	7.000%	

You are given that the annual effective 5-year spot rate is 4.669%. You are also given that the 1-year forward rate, deferred for 5 years is 8.266%. Calculate the annual effective yield on the 6-year bond.

Solution

We have:

$$s_5 = 4.669\% \quad \text{and} \quad f_5 = 8.266\%$$

We can use the BA II Plus calculator to find the price of the 5-year bond:

5 [*N*] 4.5 [*I/Y*] 6 [*PMT*] 100 [*FV*] [CPT] [*PV*]

Result is −106.5850. Price of 5-year bond is 106.5850.

We can write the price of the 5-year bond in terms of the spot rates, and therefore we can find the present value of a 5-year annuity-immediate:

$$6a_{\overline{5}|spots} + 100v_{spot}^5 = 106.5850$$
$$6a_{\overline{5}|spots} + \frac{100}{1.04669^5} = 106.5850$$
$$a_{\overline{5}|spots} = 4.4976$$

We can also write the price of the 6-year bond in terms of the spot rates. Below we use a 5-year annuity to value the first 5 coupons, and we include the final coupon with the redemption amount:

$$\text{Price of 6-year bond} = 7a_{\overline{5}|spots} + 107v_{spot}^6 = 7 \times 4.4976 + \frac{107}{(1+s_6)^6}$$
$$= 31.4832 + \frac{107}{(1+s_5)^5(1+f_5)} = 31.4832 + \frac{107}{(1.04669)^5(1.08266)}$$
$$= 110.1518$$

The BA II Plus calculator can find the yield of the 6-year bond:

6 [*N*]   −110.1518 [*PV*]   7 [*PMT*]   100 [*FV*]   [CPT] [*I/Y*]

Result is 5.000. The yield is **5.000%**.

16.04 Using Interest Rates

When we are given an interest rate, there are several questions that we must answer before we can use the interest rate in a calculation:

1. Is the interest rate a yield, spot rate, or forward rate?
2. What is the compounding frequency of the interest rate?
3. What is the time interval to which the interest rate applies?

Example 16.15

The 10-year spot rate convertible monthly is 9%. The 11-year continuously compounded spot rate is 10%.

Find the 1-year forward rate starting in 10 years, expressed as an interest rate that is compounded twice per year.

Solution

One way to obtain the answer is to convert each the rates into annual effective rates. Then, at the end, we can convert the forward rate into a rate that is compounded twice per year.

The annual effective 10-year and 11-year spot rates are:

$$s_{10} = \left(1 + \frac{0.09}{12}\right)^{12} - 1 = 0.0938$$

$$s_{11} = e^{0.10} - 1 = 0.1052$$

The annual effective forward rate is:

$$f_{10} = \frac{(1+s_{11})^{11}}{(1+s_{10})^{10}} - 1 = \frac{1.1052^{11}}{1.0938^{10}} - 1 = 0.2255$$

The forward rate, compounded twice per year, is:

$$\left(1.2255^{0.5} - 1\right) \times 2 = \mathbf{0.2141}$$

16.05 Theories on the Term Structure of Interest Rates

Yield curves are usually upward sloping, but they can sometimes be downward-sloping. A downward-sloping yield curve is also known as an inverted yield curve. If the yield curve increases and then decreases (or vice versa), then it is said to be bowed. If the yields are the same for all maturities, then the yield curve is flat. Several theories have been developed to explain the shape of the yield curve.

The **expectations theory** asserts that the term structure of interest rates reflects the expectations for future interest rates. According to the pure version of the expectations theory, the forward rates are equal to the expected values of the future spot rates. Therefore, when the term structure of interest rates is upward-sloping, interest rates are expected to increase.

The **liquidity preference theory** asserts that lenders prefer to lend for shorter lengths of time. That is, lenders have a natural preference to maintain liquidity because longer maturity bonds have longer durations, which means that they are more susceptible to price fluctuations when interest rates change. Therefore, lenders are willing to accept a lower interest rate for short-term lending, and they demand a higher interest rate for long-term lending. This theory is consistent with an upward-sloping yield curve. The liquidity preference theory is sometimes called the opportunity cost theory.

According to the **market segmentation theory**, lenders and borrowers only participate in the segment of the market that meets their borrowing or lending needs. For example, a property casualty insurance company with short-term obligations may choose to lend its funds for 6 months. A lender with long-term obligations, such as a life insurance company, may choose to lend for 20 years. The interest rate for each maturity is determined by the supply and demand for funds at that maturity. More lenders at a particular maturity leads a lower interest rate at that maturity, and more borrowers leads to a higher interest rate.

Under the market segmentation theory, a downward-sloping yield curve could be caused by:

1. Relatively high interest rates at the short end of the yield curve. This occurs when the market for short-term funds consists of:
   - a relatively small number of lenders and/or,
   - a relatively large number of borrowers.
2. Relatively low interest rates at the long end of the yield curve. This occurs when the market for long-term funds consists of:
   - a relatively large number of lenders and/or,
   - a relatively small number of borrowers.

The **preferred habitat theory** is similar to the market segmentation theory, but according to the preferred habitat theory, participants aren't necessarily restricted to a particular segment of the market. Instead, they can be persuaded to participate in another segment, provided that the interest rates are sufficiently appealing. For example, a lender that prefers to lend for 20 years may be persuaded to lend for a shorter period if the yield curve is downward-sloping and sufficiently steep.

16.06 Questions

Question 16.01

A bond matures for its par value of 1,000 in 3 years. It pays annual coupons of 5%.

The one, two, and three year annual effective spot rates are 5%, 6%, and 7%, respectively.

Calculate the price of the bond.

A 908 B 934 C 949 D 1,000 E 1,076

Question 16.02

A bond matures for its par value of 1,000 in 3 years. It pays annual coupons of 5%.

The one, two, and three year annual effective spot rates are 5%, 6%, and 7%, respectively.

Calculate the annual effective yield of the bond.

A 6.0% B 6.2% C 6.5% D 6.7% E 6.9%

Question 16.03

A bank offers certificates of deposit at the nominal annual interest rates shown below. The bank will offer the same rates for the next 6 years. The bank does not allow early withdrawals.

Term	Nominal Annual Interest Rate (Convertible Quarterly)
1 Year	3.0%
3 Years	4.0%
5 Years	4.4%

An investor deposits X now and withdraws the principal and interest at the end of 6 years. What is the maximum annual effective interest rate that the investor can earn over the 6-year period?

A 4.06% B 4.17% C 4.23% D 4.27% E 4.47%

Question 16.04

Vicki plans to invest X at the beginning of each month for 5 months. She will purchase zero-coupon bonds that mature 5 months from now. The price of each bond is shown below as a percentage of its redemption value. The prices shown below will not change over the next 5 months.

Maturity (in months)	1	2	3	4	5
Price	99%	98%	97%	96%	95%

Calculate the value of X that results in an accumulated value of 10,000 at the end of 5 months.

A 1,923 B 1,940 C 1,948 D 1,963 E 1,990

Question 16.05

Annual effective yields for zero-coupon bonds are currently quoted at 7.5% for one-year bonds, 8.5% for two-year bonds, and 9.25% for three-year bonds.

Let *i* be the one-year forward rate, starting in 1 year.

Calculate *i*.

A 8.0% B 9.5% C 10.0% D 10.8% E 11.0%

Question 16.06

The prices of three zero-coupon are shown in the table below. The maturity value of each of the bonds is 1.

Maturity	Price
1 Year	0.952
2 Years	0.920
3 Years	0.850

Calculate the 1-year forward rate, deferred for 2 years.

A 3.5% B 7.0% C 7.6% D 8.2% E 12.0%

Question 16.07

You are given the following term structure of interest rates.

Years	Annual Effective Spot Rate
1	6.0%
2	7.5%
3	8.0%
4	8.5%
5	9.0%
6	9.5%

Calculate the 1-year forward rate, starting in 5 years.

A 3.5% B 7.0% C 7.6% D 8.2% E 12.0%

Question 16.08

You are given the following term structure of interest rates.

Years	Annual Effective Spot Rate
1	6.0%
2	7.5%
3	8.0%
4	8.5%
5	9.0%
6	9.5%

In 3 years, a lender will loan 1,000 for 1 year. The loan will be repaid 4 years from today with a single payment of X. The interest rate on the loan is locked in today.

Calculate X.

A 1,075 B 1,080 C 1,085 D 1,090 E 1,100

Question 16.09

You are given the following term structure of interest rates.

Years	Annual Effective Spot Rate
1	6.0%
2	7.5%
3	8.0%
4	8.5%
5	9.0%
6	9.5%

In 3 years, a lender will loan 1,000 for 2 years. The loan will be repaid 5 years from today with a single payment of X. Using the appropriate forward rates, the interest rate on the loan is locked in today.

Calculate X.

A 1,166 B 1,183 C 1,188 D 1,221 E 1,232

Question 16.10

The one-year forward rates, deferred for t years, are described in the table below:

t	0	1	2	3	4
One-Year Forward Rate	5%	3%	2%	4%	6%

Calculate the spot rate for a bond that matures three years from now.

A 3.0% B 3.3% C 4.0% D 6.0% E 8.1%

Question 16.11

Which of the following statements are true?

I. If a par value bond has a price that is equal to its par value, then it has a yield that is equal to its coupon rate.

II. An n-year spot rate is greater than or equal to the yield of an n-year bond.

III. An n-year spot rate is less than or equal to the 1-year forward rate deferred for $(n - 1)$ years.

A I only

B II only

C III Only

D II and III

E The correct answer is not given by (A), (B), (C), or (D).

Question 16.12

A yield curve is defined by the following equation:

$$i_t = 0.02 - 0.001t + 0.001t^2$$

where i_t is the annual effective rate of return for a zero-coupon bond that matures in t years.

Let j be the one-year forward rate, deferred for 4 years.

Calculate j.

A 3.14% B 4.00% C 7.26% D 9.10% E 12.52%

Question 16.13

You are given the following term structure of interest rates.

Years	Annual Effective Spot Rate
1	3.0%
2	3.5%
3	3.9%
4	4.4%
5	5.2%

In one year, a 3-year annuity-immediate will be issued. The annuity makes annual payments of 10,000.

Calculate the present value of this annuity one year from now.

A 25,847 B 26,669 C 27,469 D 27,960 E 28,798

Question 16.14

A company must pay liabilities of 3,000 in 1 year and 5,000 in 3 years.

There are two zero-coupon bonds available for investment:

Bond A: One-year bond with an annual effective yield of 4%.

Bond B: Two-year bond with an annual effective yield of 5%.

The one-year forward rate, starting in 2 years, is 5.5%.

Calculate the cost to create an investment portfolio that matches the liabilities exactly.

A 7,183 B 7,225 C 7,420 D 7,450 E 7,463

Question 16.15

The yield curve was previously flat, but it is now inverted. Which of the following is a reasonable explanation for the downward slope of the yield curve?

A An increase in the pool of lenders that desire to lend for a short period of time.

B A decrease in the pool of borrowers that desire to borrow for a short period of time.

C A decrease in the pool of lenders that desire to lend for a long period of time.

D A decrease in the pool of borrowers that desire to borrow for a long period of time.

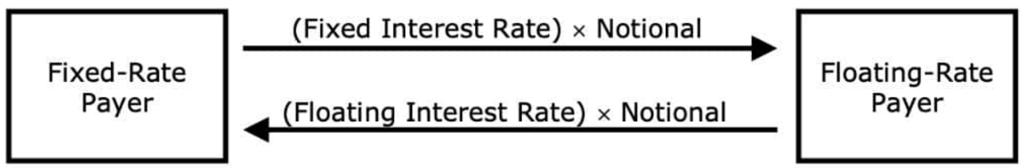
E The liquidity preference theory has become a more accurate description of investor preferences.

Chapter 17: Interest Rate Swaps

An **interest rate swap** is based on a notional amount of principal and has two counterparties:

- One counterparty pays a fixed rate of interest and receives a floating rate. This counterparty is the **fixed-rate payer**, and the fixed rate of interest is the **swap rate**.
- The other counterparty pays the floating rate of interest and receives a fixed rate of interest. This counterparty is the **floating-rate payer**.

The interest rates paid by the two counterparties are multiplied by a **notional** amount.



The dates on which the payments are made are called the settlement dates, and the length of time between two settlement dates is the **settlement period**. The settlement period is usually a constant amount, such as one year.

*The fixed-rate payer is sometimes simply called the **payer**, and the floating-rate payer is then referred to as the **receiver**. That is, the payer pays the swap rate and the receiver receives the swap rate.*

17.01 Establishing the Swap Rate

Consider two hypothetical bonds that have the same time until maturity, n, as an interest rate swap. This time until maturity is called the **swap term** (also known as the swap tenor). The two bonds also have par values that are equal to the notional amount of the interest rate swap, and they both make coupon payments at the same times:

1. The first bond is a fixed-rate bond with a coupon rate that is equal to the swap rate:

   $F = \text{Notional amount of swap}$

   $c = \text{Fixed rate of swap} = \text{Swap rate}$

2. The second bond is a floating-rate bond, also known as a variable-rate bond. At the beginning of each coupon period, the floating-rate bond's coupon rate is reset to the one-period spot rate applicable at that time. Therefore, at the beginning of each coupon period, the price of the floating-rate bond is equal to its notional amount:

   $F = \text{Price of floating-rate bond at beginning of coupon period}$

The net cash flows of the two counterparties to the interest rate swap could be replicated as follows:

- The fixed-rate payer could issue the fixed-rate bond and buy the floating-rate bond from the floating-rate payer.
- The floating-rate payer could issue the floating-rate bond and buy the fixed-rate bond from the fixed-rate payer.

Since both bonds pay the par value upon maturity, the par value payments cancel each other out, and there is no need to actually make those payments. Therefore, only the net coupon payments are made. If the floating rate is higher than the swap rate, then the floating-rate payer pays the net payment and the fixed-rate payer receives the net payment, and vice versa:

$$i_{float} > c \quad \Rightarrow \quad \text{Net Payment to fixed-rate payer} = (i_{float} - c)F$$

$$i_{float} < c \quad \Rightarrow \quad \text{Net Payment to floating-rate payer} = (c - i_{float})F$$

where:

$$i_{float} = \text{Floating interest rate at beginning of coupon period}$$

When an interest rate swap is originated, there is no exchange of funds between the two counterparties. This is because the fixed rate is set so that the future payments of the hypothetical fixed-rate bond have the same present value as the future payments of the hypothetical floating-rate bond. This fixed rate is equal to the swap rate of the interest rate swap.

The price of the floating-rate bond is equal to its face amount, and the price of the fixed-rate bond is equal to the present value of its fixed cash flows. Setting these two amounts equal to one another allows us to solve for the coupon rate of the fixed-rate bond, which is equal to the yield of a par bond. This par bond yield is equal to the swap rate of the interest rate swap:

$$\text{Price(Floating-rate bond)} = \text{Price(Fixed-rate bond)}$$

$$F = Fc \times a_{\overline{n}|y} + \frac{F}{(1+y)^n}$$

$$1 = c \times a_{\overline{n}|y} + \frac{1}{(1+y)^n}$$

$$1 = c\,\frac{1-(1+y)^{-n}}{y} + \frac{1}{(1+y)^n}$$

$$(1+y)^n = \frac{c}{y}\left[(1+y)^n - 1\right] + 1$$

$$(1+y)^n - 1 = \frac{c}{y}\left[(1+y)^n - 1\right]$$

$$c = y$$

The formula in the Key Concept below is the formula for the par bond yield that we saw in Section 16.02.

17.01

Swap Rate

When an interest rate swap is originated, the swap rate is equal to the yield of a par bond, which is also equal to the coupon rate of the par bond:

$$c = \frac{1 - v_{spot}^n}{a_{\overline{n}|spots}} = \frac{1 - P_n}{P_1 + P_2 + \cdots + P_n} \qquad \text{where: } n = \text{swap term}$$

The example below shows how an interest rate swap's net payments are calculated and when those payments occur.

Example 17.01

An interest rate swap has the following swap rate and swap term:

Swap rate = 4.0%

Swap term = 3 years

The floating rate is the 1-year spot rate, and the notional amount is \$100,000.

The current 1-year spot rate is 2%.

One year later, the 1-year spot rate is revealed to be 3%.

Two years later, the 1-year spot rate is revealed to be 6.5%.

Calculate the net cash flows at the end of 1, 2, and 3 years and determine who pays whom.

Solution At the outset of the swap, the swap rate is greater than the floating rate, so the fixed-rate payer makes the following payment to the floating-rate payer at the end of the first year:

$$\text{Net Payment to floating-rate payer} = (0.04 - 0.02)100{,}000 = \mathbf{2{,}000}$$

The second payment is made at the end of the second year, and it is based on the spot rate that becomes known at the end of the first year:

$$\text{Net Payment to floating-rate payer} = (0.04 - 0.03)100{,}000 = \mathbf{1{,}000}$$

The third payment is made at the end of the third year, and it is based on the spot rate that becomes known at the end of the second year. Since the floating rate is greater than the swap rate, the net payment is from the floating-rate payer and is made to the fixed-rate payer:

$$\text{Net Payment to fixed-rate payer} = (0.065 - 0.04)100{,}000 = \mathbf{2{,}500}$$

The example below shows how the swap rate is calculated.

Example 17.02 The term structure of interest rates is given below:

Maturity	Annual Effective Spot Rate
1	3.0%
2	4.0%
3	5.0%

The notional amount of a newly created 3-year interest rate swap is $1,000,000.

a. Calculate the swap rate.

b. One year later, the new 1-year spot rate is 5.5%. Calculate the net amount paid at time 2 by the floating-rate payer to the fixed-rate payer.

Solution a. The swap rate is:

$$c = \frac{1 - v_{spot}^{n}}{a_{\overline{n|}spots}} = \frac{1 - v_{spot}^{3}}{a_{\overline{3|}spots}} = \frac{1 - (1.05)^{-3}}{\frac{1}{1.03} + \frac{1}{1.04^2} + \frac{1}{1.05^3}} = \mathbf{0.049347}$$

b. The 1-year spot is the floating rate in the expression below:

$$i_{float} > c \quad \Rightarrow \quad \text{Net Payment to fixed-rate payer} = (i_{float} - c)F$$

The new 1-year spot rate applies from time 1 to time 2, and therefore the payment associated with it is made at time 2. At the end of the second year, the floating-rate payer pays the difference between the floating rate and the swap rate to the fixed-rate payer:

$$\text{Net Payment to fixed-rate payer} = (0.055 - 0.049347)1{,}000{,}000 = \mathbf{5{,}652.70}$$

In the example below, we solve for the value of an annuity that is based on the spot rates. We then use the value of that annuity to find the swap rate.

Example 17.03 A 9-year bond pays annual coupons of 6.5% per year. It is priced to have an annual effective yield of 6.75%. The annual effective 9-year spot rate is 7.359%.

Calculate the 9-year swap rate.

Solution We can use the BA II Plus calculator to find the price of the 9-year bond per $100 of par value:

9 [*N*] 6.75 [*I/Y*] 6.5 [*PMT*] 100 [*FV*] [CPT] [*PV*]

Result is −98.3537. Price of 9-year bond is 98.3537.

We can write the price of the 9-year bond in terms of the spot rates, and therefore we can find the present value of a 9-year annuity-immediate:

$$6.50 \times a_{\overline{9}|spots} + 100v_{spot}^{9} = 98.3537$$

$$6.50a_{\overline{9}|spots} + \frac{100}{1.07359^9} = 98.3537$$

$$a_{\overline{9}|spots} = 7.0116$$

The coupon of a 9-year bond that is priced at par is:

$$c = \frac{1 - v_{spot}^{n}}{a_{\overline{n}|spots}} = \frac{1 - 1.07359^{-9}}{7.0116} = \mathbf{0.06735}$$

17.02 Valuing Interest Rate Swaps

At the outset, the value of the interest rate swap to both counterparties is zero because both the hypothetical floating-rate bond and the hypothetical fixed-rate bond are priced at par. As time passes, the hypothetical floating-rate bond continues to be priced at par, but the fixed-rate bond's payments do not change as interest rates change, and therefore the fixed-rate bond's price is not maintained at par.

The fixed-rate payer effectively owns a floating-rate bond and makes the payments of a fixed-rate bond. Therefore just after an interest payment made at time t, the value of the interest rate swap to the fixed-rate payer is the value of the floating-rate bond, which is equal to the notional amount F, minus the value of the remaining payments from the fixed-rate bond:

$$\text{Value to fixed-rate payer} = F - PV_t(\text{Fixed rate bond})$$

If, at time t, the new swap rate is s for an interest rate swap that matures at time n, then the hypothetical fixed-rate bond with a coupon of s is priced at par:

$$F = \frac{sF}{(1+s_1)} + \frac{sF}{(1+s_2)^2} + \cdots + \frac{sF}{(1+s_{n-t})^{n-t}} + \frac{F}{(1+s_{n-t})^{n-t}}$$

The original interest rate swap, entered into at time 0, still has a fixed rate of c, so its value to the fixed-rate payer is:

$$\text{Value to fixed-rate payer} = F - PV_t(\text{Fixed rate bond})$$

$$= F - \left[\frac{cF}{(1+s_1)} + \frac{cF}{(1+s_2)^2} + \cdots + \frac{cF}{(1+s_{n-t})^{n-t}} + \frac{F}{(1+s_{n-t})^{n-t}}\right]$$

$$= \left[\frac{sF}{(1+s_1)} + \frac{sF}{(1+s_2)^2} + \cdots + \frac{sF}{(1+s_{n-t})^{n-t}} + \frac{F}{(1+s_{n-t})^{n-t}}\right]$$

$$- \left[\frac{cF}{(1+s_1)} + \frac{cF}{(1+s_2)^2} + \cdots + \frac{cF}{(1+s_{n-t})^{n-t}} + \frac{F}{(1+s_{n-t})^{n-t}}\right]$$

$$= \frac{(s-c)F}{(1+s_1)} + \frac{(s-c)F}{(1+s_2)^2} + \cdots + \frac{(s-c)F}{(1+s_{n-t})^{n-t}} = (s-c)F \times a_{\overline{n-t}|spots}$$

The final expression above tells us that the value of the swap to the fixed-rate payer is the present value of the differences between the fixed-rate payments of a new swap and the fixed-rate payments of the older swap. The value of the swap to the floating-rate payer is just the opposite:

$$\text{Value to floating-rate payer} = PV_t(\text{Fixed-rate bond}) - F = (c-s)F \times a_{\overline{n-t}|spots}$$

As we would expect, when the swap rate increases, the value of an existing interest rate swap increases for the fixed-rate payer and decreases for the floating-rate payer.

Valuing Interest Rate Swaps

17.02 At time t, the value of the interest rate swap is the difference between the notional amount of the swap and the present value of the remaining fixed payments of the hypothetical fixed-rate bond:

$$\text{Value to fixed-rate payer} = F - PV_t(\text{Fixed-rate bond})$$
$$\text{Value to floating-rate payer} = PV_t(\text{Fixed-rate bond}) - F$$

In the example below, the interest rates have increased after the swap originated. Since the fixed-rate payer receives floating rate payments, the future payments will be higher, making the interest rate swap more valuable to the fixed-rate payer.

Example 17.04 A three-year interest rate swap has a swap rate of 4.9347% and a notional amount of \$1,000,000. One year after the origination of the swap, when there are only two payments remaining, the 1-year spot rate is 7% and the 2-year spot rate is 8%.

Calculate the current value of the interest rate swap to:

a. The fixed-rate payer, and
b. The floating-rate payer.

Solution a. The value of the swap to the fixed-rate payer is:

$$\text{Value to fixed-rate payer} = F - PV_t(\text{Fixed-rate bond})$$
$$= 1{,}000{,}000 - \left[\frac{49{,}347}{1.07} + \frac{1{,}049{,}347}{1.08^2}\right] = \mathbf{54{,}235.39}$$

b. The value of the swap to the floating-rate payer is the opposite of the value to the fixed-rate payer:

$$\text{Value to floating-rate payer} = PV_t(\text{Fixed-rate bond}) - F = -\mathbf{54{,}235.39}$$

In the example below, we solve for the value of an annuity that is based on the spot rates. We then use that value of that annuity to find the current value of the interest rate swap.

Example 17.05 A 9-year interest rate swap has a notional amount of \$100,000 and a fixed swap rate of 6.735%. The swap makes payments annually.

After 2-years:

1. The interest rate swap is a 7-year swap.
2. A 7-year bond with annual coupons of 4% and par of \$100 has a price of \$85.70.
3. The 7-year spot rate is 7%.

Find the value of the swap to the floating-rate payer at the end of the 2 years.

Solution We can write the price of the 7-year bond in terms of the spot rates, and therefore we can find the present value of a 7-year annuity-immediate:

$$4 \times a_{\overline{7}|spots} + 100v_{spot}^7 = 85.70$$
$$4a_{\overline{7}|spots} + \frac{100}{1.07^7} = 85.70$$
$$a_{\overline{7}|spots} = 5.8563$$

The price of the hypothetical fixed-rate bond with a coupon rate equal to the swap rate is:

$$6{,}735a_{\overline{7}|spots} + 100{,}000v_{spot}^7 = 6{,}735 \times 5.8563 + \frac{100{,}000}{1.07^7} = 101{,}716.86$$

The value of the swap to the floating-rate payer is:

$$\text{Value to floating-rate payer} = PV_t(\text{Fixed-rate bond}) - F$$
$$= 101{,}716.86 - 100{,}000 = \mathbf{1{,}716.86}$$

Alternatively, we can find the new swap rate:

$$s = \frac{1 - v_{spot}^7}{a_{\overline{7}|spots}} = \frac{1-(1.07)^{-7}}{5.8563} = 0.064418$$

The value of the swap to the floating rate payer is:

$$\text{Value to floating-rate payer} = (c - s)F \times a_{\overline{n-t}|spots}$$
$$= (0.06735 - 0.064418) \times 100{,}000 \times 5.8563 = \mathbf{1{,}716.86}$$

17.03 Forward Rates and Interest Rate Swaps

If a borrower issues a bond that locks in the current forward rates, then the bond is issued at par:

$$F = F \times \left[f_0 \times v_{spot}^1 + f_1 \times v_{spot}^2 + \cdots + f_{n-1} \times v_{spot}^n \right] + F \times v_{spot}^n$$

This suggests another way to calculate the swap rate for an interest rate swap. The swap rate must cause the value of the fixed-rate bond to be equal to par, and we can write the par value in terms of the forward rates:

$$\text{Price(Fixed-rate bond)} = \text{Price(Floating-rate bond)}$$
$$Fc\left[v_{spot}^1 + v_{spot}^2 + \cdots + v_{spot}^n\right] + Fv_{spot}^n = F$$
$$Fc\left[v_{spot}^1 + v_{spot}^2 + \cdots + v_{spot}^n\right] + Fv_{spot}^n$$
$$= F \times \left[f_0 \times v_{spot}^1 + f_1 \times v_{spot}^2 + \cdots + f_{n-1} \times v_{spot}^n \right] + F \times v_{spot}^n$$
$$c\left[v_{spot}^1 + v_{spot}^2 + \cdots + v_{spot}^n\right] = f_0 \times v_{spot}^1 + f_1 \times v_{spot}^2 + \cdots + f_{n-1} \times v_{spot}^n$$
$$c \times \sum_{j=1}^{n}\left(v_{spot}^j\right) = \sum_{j=1}^{n}\left(f_{j-1} \times v_{spot}^j\right)$$
$$c = \frac{\sum_{j=1}^{n}\left(f_{j-1} \times v_{spot}^j\right)}{\sum_{j=1}^{n}\left(v_{spot}^j\right)}$$

The expression above shows that the swap rate is the weighted average of the forward rates. The weights are the present value factors.

17.03

Swap Rate as a Weighted Average of Forward Rates

The swap rate of an interest rate swap is a weighted average of the forward rates, where the weights are the present value factors:

$$c = \frac{\sum_{j=1}^{n}\left(f_{j-1} \times v_{spot}^j\right)}{\sum_{j=1}^{n}\left(v_{spot}^j\right)}$$

Another way to interpret the formula above is to observe that the present value of the coupon payments of a par bond must be equal to the present value of the interest payments that can be locked in using the forward rates:

$$c \times \sum_{j=1}^{n}\left(v_{spot}^{j}\right) = \sum_{j=1}^{n}\left(f_{j-1} \times v_{spot}^{j}\right)$$

The example below uses the Key Concept above to answer the first part of the question that we originally saw in Example 17.02.

Example 17.06

The term structure of interest rates is given below:

Maturity	Annual Effective Spot Rate
1	3.0%
2	4.0%
3	5.0%

The notional amount of a newly created 3-year interest rate swap is $1,000,000.

Calculate the swap rate.

Solution

The forward rates are:

$$f_0 = 0.03$$

$$f_1 = \frac{1.04^2}{1.03} - 1 = 0.05001$$

$$f_2 = \frac{1.05^3}{1.04^2} - 1 = 0.07029$$

The swap rate is the weighted average of the forward rates:

$$c = \frac{\sum_{j=1}^{n}\left(f_{j-1} \times v_{spot}^{j}\right)}{\sum_{j=1}^{n}\left(v_{spot}^{j}\right)} = \frac{\frac{0.03}{1.03} + \frac{0.05001}{1.04^2} + \frac{0.07029}{1.05^3}}{\frac{1}{1.03} + \frac{1}{1.04^2} + \frac{1}{1.05^3}} = \frac{0.13616}{2.75927} = \mathbf{0.049347}$$

17.04 Deferred Interest Rate Swaps

A deferred swap is similar to a regular swap, but the payments are deferred for an interval of time.

We again consider a fixed-rate bond and a floating-rate bond, but now the payments are deferred for k units of time. The first payment occurs at time $(k + 1)$, and it is based on the floating rate that is revealed at time k.

At time k, the floating-rate bond will be priced at par. We can use the spot rates available now to solve for the coupon rate of the fixed-rate bond that causes the value of the fixed-rate bond to be equal to the value of the floating-rate bond. This coupon rate is the swap rate of the deferred swap:

$$\text{Price(Floating-rate bond)} = \text{Price(Fixed-rate bond)}$$

$$Fv_{spot}^{k} = Fc \times \left(v_{spot}^{k+1} + v_{spot}^{k+2} + \cdots + v_{spot}^{n}\right) + Fv_{spot}^{n}$$

$$Fv_{spot}^{k} = Fc \times \left(a_{\overline{n|}spots} - a_{\overline{k|}spots}\right) + Fv_{spot}^{n}$$

$$v_{spot}^{k} = c \times \left(a_{\overline{n|}spots} - a_{\overline{k|}spots}\right) + v_{spot}^{n}$$

$$c = \frac{v_{spot}^{k} - v_{spot}^{n}}{a_{\overline{n|}spots} - a_{\overline{k|}spots}}$$

We can express the denominator of the final expression above in terms of the discount factors:

$$c = \frac{v_{spot}^k - v_{spot}^n}{a_{\overline{n}|spots} - a_{\overline{k}|spots}} = \frac{v_{spot}^k - v_{spot}^n}{v_{spot}^{k+1} + v_{spot}^{k+2} + \cdots + v_{spot}^n}$$

17.04

Deferred Interest Rate Swaps

The swap rate of an interest rate swap that matures at time n, is deferred for k units of time, and makes $(n-k)$ payments is:

$$c = \frac{v_{spot}^k - v_{spot}^n}{v_{spot}^{k+1} + v_{spot}^{k+2} + \cdots + v_{spot}^n} = \frac{P_k - P_n}{P_{k+1} + P_{k+2} + \cdots + P_n}$$

If the swap is deferred for 0 units of time, then $k = 0$, and the swap is a regular interest rate swap. Not surprisingly, the swap rate is then the same as swap rate found in Section 17.01:

$$c = \frac{v_{spot}^k - v_{spot}^n}{v_{spot}^{k+1} + v_{spot}^{k+2} + \cdots + v_{spot}^n} = \frac{v_{spot}^0 - v_{spot}^n}{v_{spot}^1 + v_{spot}^2 + \cdots + v_{spot}^n} = \frac{1 - v_{spot}^n}{a_{\overline{n}|spots} - 0} = \frac{1 - v_{spot}^n}{a_{\overline{n}|spots}}$$

Example 17.07

The spot rates are given below.

Maturity	Annual Effective Spot Rate
1	3.0%
2	4.0%
3	5.0%
4	5.5%
5	6.0%

A 3-year interest rate swap is deferred for 2 years, so its payments are scheduled to occur at the end of years 3, 4, and 5.

Calculate the swap rate for the 3-year swap that is deferred for 2 years.

Solution

The swap matures at time 5, and its payments are deferred for 2 years, so it will make 3 payments. The payments occur at the end of 3 years, 4 years, and 5 years. The swap rate of the deferred swap is:

$$c = \frac{v_{spot}^k - v_{spot}^n}{v_{spot}^{k+1} + v_{spot}^{k+2} + \cdots + v_{spot}^n} = \frac{v_{spot}^2 - v_{spot}^5}{v_{spot}^3 + v_{spot}^4 + v_{spot}^5} = \frac{\dfrac{1}{1.04^2} - \dfrac{1}{1.06^5}}{\dfrac{1}{1.05^3} + \dfrac{1}{1.055^4} + \dfrac{1}{1.06^5}}$$

$$= \frac{0.17730}{2.41831} = \mathbf{0.073315}$$

The initial value of a deferred swap is zero, but after it is issued, its value may be positive or negative.

To find the value of a deferred swap, we again make use of the fact that the value of the floating-rate bond will have a value of F one unit of time before the first payment is made. That is, the floating-rate bond will have a value of F at time k. The value of the fixed-rate bond is found by calculating the present value of the fixed payments.

The value to the fixed-rate payer is the opposite of the value to the floating-rate payer:

Value to fixed-rate payer = PV_t(Floating-rate bond) – PV_t(Fixed-rate bond)

Value to floating-rate payer = PV_t(Fixed-rate bond) – PV_t(Floating-rate bond)

Example 17.08

A 3-year swap was established with a swap rate of 7.3315% and a notional amount of \$100,000. The swap was a 2-year deferred swap, so its payments were scheduled to be made 3, 4, and 5 years after the swap originated.

A year has elapsed and the new spot curve is shown below.

Maturity	Annual Effective Spot Rate
1	4.0%
2	4.5%
3	5.0%
4	6.0%

Calculate the current value of the swap to the floating-rate payer.

Solution

One year after the origination of the swap, the remaining payments occur in 2, 3, and 4 years. Therefore, the payments of the hypothetical fixed-rate bond occur in 2, 3, and 4 years. The value of the floating-rate bond will be \$100,000 in 1 year, and we can find its present value now by discounting at the 1-year spot rate:

$$\text{Value to floating-rate payer} = PV_t(\text{Fixed-rate bond}) - PV_t(\text{Floating-rate bond})$$

$$= \left[\frac{7{,}331.50}{1.045^2} + \frac{7{,}331.50}{1.05^3} + \frac{107{,}331.50}{1.06^4}\right] - \frac{100{,}000}{1.04}$$

$$= 98{,}063.5005 - 96{,}153.8462 = \mathbf{1{,}909.65}$$

17.05 Swaps With Varying Notional Amount

Interest rate swaps can be structured so that the notional amount varies over time. If the notional amount declines, then the swap is known as an **amortizing swap**, and if the notional amount increases, then the swap is known as an **accreting swap**.

Let F_j be the notional amount that determines the j^{th} payment of an interest rate swap.

Upon origination, the present value of the fixed-rate payments is equal to the present value of the interest rate payments that could be locked in using the forward rates:

$$c\left[F_1 \times v_{spot}^1 + F_2 \times v_{spot}^2 + \cdots + F_n \times v_{spot}^n\right]$$

$$= F_1 \times f_0 \times v_{spot}^1 + F_s \times f_1 \times v_{spot}^2 + \cdots + F_n \times f_{n-1} \times v_{spot}^n$$

$$c \times \sum_{j=1}^{n}\left(F_j \times v_{spot}^j\right) = \sum_{j=1}^{n}\left(F_j \times v_{spot}^j \times f_{j-1}\right)$$

$$c = \frac{\sum_{j=1}^{n}\left(F_j \times v_{spot}^j \times f_{j-1}\right)}{\sum_{j=1}^{n}\left(F_j \times v_{spot}^j\right)}$$

The swap rate is still a weighted average, but now the weights are the present values of the notional amounts.

17.05

Swap Rate When Notional Amount Varies

The swap rate of an interest rate swap with a varying notional amount is a weighted average of the forward rates, where the weights are the present values of the notional amounts:

$$c = \frac{\sum_{j=1}^{n}\left(F_j \times v_{spot}^{j} \times f_{j-1}\right)}{\sum_{j=1}^{n}\left(F_j \times v_{spot}^{j}\right)}$$

The swap in the example below has a notional amount that declines over time.

Example 17.09

The forward rates are given below.

t	Annual Effective Forward Rate f_t
0	2.0%
1	3.5%
2	4.0%

An interest rate swap makes payments at the end of 1, 2, and 3 years. The first payment is based on a notional amount of 100,000, the second payment is based on a notional amount of 75,000, and the third payment is based on a notional amount of 50,000.

Calculate the swap rate of the interest rate swap.

Solution

The swap rate is:

$$c = \frac{\sum_{j=1}^{n}\left(F_j \times v_{spot}^{j} \times f_{j-1}\right)}{\sum_{j=1}^{n}\left(F_j \times v_{spot}^{j}\right)}$$

$$= \frac{100{,}000 \times \frac{0.02}{1.02} + 75{,}000 \times \frac{0.035}{1.02 \times 1.035} + 50{,}000 \times \frac{0.04}{1.02 \times 1.035 \times 1.04}}{100{,}000 \times \frac{1}{1.02} + 75{,}000 \times \frac{1}{1.02 \times 1.035} + 50{,}000 \times \frac{1}{1.02 \times 1.035 \times 1.04}}$$

$$= \frac{6{,}268.8992}{214{,}622.4525} = \mathbf{0.029209}$$

We can use the Key Concept above to find the swap rate of a deferred swap. This is accomplished by setting the notional amount to zero in the years during which the swap is deferred. The example below examines the same question that was addressed above in Example 17.07.

Example 17.10

The spot rates are given below.

Maturity	Annual Effective Spot Rate
1	3.0%
2	4.0%
3	5.0%
4	5.5%
5	6.0%

A 3-year interest rate swap is deferred for 2 years, so its payments are scheduled to occur at the end of years 3, 4, and 5.

Calculate the swap rate for the 3-year swap that is deferred for 2 years.

Solution The forward rates are in the 3rd, 4th, and 5th years are:

$$f_2 = \frac{1.05^3}{1.04^2} - 1 = 0.070289$$

$$f_3 = \frac{1.055^4}{1.05^3} - 1 = 0.0701433$$

$$f_4 = \frac{1.06^5}{1.055^4} - 1 = 0.080238$$

The swap matures at time 5, and its payments are deferred for 2 years, so it will make 3 payments. The payments occur at the end of 3 years, 4 years, and 5 years. Therefore, we can treat the swap as having zero notional amount for the first two years. It doesn't matter what notional amount we assume for the final 3 years, as long as the assumed amount is held constant. For convenience, let's assume that the notional amount in the final 3 years is 1.

The swap rate of the deferred swap is:

$$c = \frac{\sum_{j=1}^{n}\left(F_j \times v_{spot}^j \times f_{j-1}\right)}{\sum_{j=1}^{n}\left(F_j \times v_{spot}^j\right)} = \frac{\frac{0 \times f_0}{1.03^1} + \frac{0 \times f_1}{1.04^2} + \frac{1 \times f_2}{1.05^3} + \frac{1 \times f_3}{1.055^4} + \frac{1 \times f_4}{1.06^5}}{\frac{0}{1.03^1} + \frac{0}{1.04^2} + \frac{1}{1.05^3} + \frac{1}{1.055^4} + \frac{1}{1.06^5}}$$

$$= \frac{\frac{1 \times 0.070289}{1.05^3} + \frac{1 \times 0.0701433}{1.055^4} + \frac{1 \times 0.080238}{1.06^5}}{\frac{1}{1.05^3} + \frac{1}{1.055^4} + \frac{1}{1.06^5}} = \frac{0.177298}{2.418313} = \mathbf{0.073315}$$

Previously, when finding the value of a swap that was issued earlier, we noted that the value of the hypothetical floating-rate bond was equal to the notional amount. When the notional amount varies over time, however, we use the forward rates to find the value of the payments made by the floating-rate payer:

$$\text{Value to fixed-rate payer} = PV_t(\text{Floating-rate bond}) - PV_t(\text{Fixed-rate bond})$$

$$= \left[\sum_{j=1}^{n}\left(F_j \times v_{spot}^j \times f_{j-1}\right) + F_n \times v_{spot}^n\right] - c \times \left[\sum_{j=1}^{n}\left(F_j \times v_{spot}^j\right) + F_n \times v_{spot}^n\right]$$

$$= \sum_{j=1}^{n}\left(F_j \times v_{spot}^j \times f_{j-1}\right) - c \times \sum_{j=1}^{n}\left(F_j \times v_{spot}^j\right)$$

$$= \sum_{j=1}^{n}\left(F_j \times v_{spot}^j \times \left[f_{j-1} - c\right]\right)$$

The value to the floating-rate payer is the opposite of the value to the fixed-rate payer.

17.06

Valuing Swaps With Varying Notional Amount

The value of an interest rate swap is the present value of the difference between the payments based on the forward rates and the payments based on the swap rate:

$$\text{Value to fixed-rate payer} = \sum_{j=1}^{n}\left(F_j \times v_{spot}^j \times \left[f_{j-1} - c\right]\right)$$

$$\text{Value to floating-rate payer} = \sum_{j=1}^{n}\left(F_j \times v_{spot}^j \times \left[c - f_{j-1}\right]\right)$$

In the example below the swap rate is less than or equal to the forward rates, so the swap value to the fixed-rate payer is positive.

Example 17.11

The forward rates are given below.

t	Annual Effective Forward Rate f_t
0	2.0%
1	3.5%
2	4.0%

An interest rate swap makes payments at the end of 1, 2, and 3 years. The first payment is based on a notional amount of 100,000, the second payment is based on a notional amount of 75,000, and the third payment is based on a notional amount of 50,000.

The swap rate is 2.0%. Calculate the value of the interest rate swap to the fixed-rate payer.

Solution

The swap rate is:

$$\text{Value to fixed-rate payer} = \sum_{j=1}^{n}\left(F_j \times v_{spot}^{j} \times \left[f_{j-1} - c\right]\right)$$

$$= 100{,}000 \times \frac{0.02 - 0.02}{1.02} + 75{,}000 \times \frac{0.035 - 0.02}{1.02 \times 1.035} + 50{,}000 \times \frac{0.04 - 0.02}{1.02 \times 1.035 \times 1.04}$$

$$= \mathbf{1{,}976.45}$$

Let's use the Key Concept above to answer the question posed in Example 17.08.

Example 17.12

A 3-year swap was established with a swap rate of 7.3315% and a notional amount of \$100,000. The swap was a 2-year deferred swap, so its payments were scheduled to be made 3, 4, and 5 years after the swap originated.

A year has elapsed and the new spot curve is shown below.

Maturity	Annual Effective Spot Rate
1	4.0%
2	4.5%
3	5.0%
4	6.0%

Calculate the current value of the swap to the floating-rate payer.

Solution

One year after the origination of the swap, the remaining payments occur in 2, 3, and 4 years. Therefore, the payments of the hypothetical bonds occur in 2, 3, and 4 years.

The forward interest rates needed to answer this question are shown below:

$$f_1 = \frac{1.045^2}{1.04} - 1 = 0.05002$$

$$f_2 = \frac{1.05^3}{1.045^2} - 1 = 0.06007$$

$$f_3 = \frac{1.06^4}{1.05^3} - 1 = 0.09058$$

We can set the notional amount equal to zero for one year, when $j = 1$ below, to find the value of the swap to the floating rate payer:

$$\text{Value to floating-rate payer} = \sum_{j=1}^{n}\left(F_j \times v_{spot}^{j} \times \left[c - f_{j-1}\right]\right)$$

$$= \frac{100,000 \times (0.073315 - f_2)}{1.045^2} + \frac{100,000 \times (0.073315 - f_3)}{1.05^3} + \frac{100,000 \times (0.073315 - f_4)}{1.06^4}$$

$$= 100,000\left(\frac{0.073315 - 0.05002}{1.045^2} + \frac{0.073315 - 0.06007}{1.05^3} + \frac{0.073315 - 0.09058}{1.06^4}\right)$$

$$= 100,000\left(0.02133 + 0.01144 - 0.01367\right) = \mathbf{1,909.65}$$

17.06 Benchmark Interest Rates for Interest Rate Swaps

The floating rate most often used as a benchmark for dollar-denominated interest rate swaps is the London Interbank Offered Rate (LIBOR). This interest rate is an average of the interest rates that large banks charge one another when lending dollars between banks. The LIBOR rate is available for a range of maturities, from one day to one year. The LIBOR rates are published every business day.

Another common benchmark for dollar-denominated interest rate swaps is an index of the prime interest rates charged by banks when loaning funds to their best customers. The Wall Street Journal publishes a Prime Rate index, which is an average of the prime rates of the largest banks in the United States. The prime rate is usually about 300 basis points greater than the Fed Funds rate.

Interest rate swaps denominated in euros are usually based on the Euro Interbank Offered Rate (EURIBOR), which is the average of the interest rates that large banks charge one another when lending euros.

Floating rate payments are commonly based on an index plus a spread. For example, a variable-rate bond might pay 1-year LIBOR plus 200 basis points.

17.07 Applications of Interest Rate Swaps

Companies use interest rate swaps to adjust their exposure to changes in interest rates. A borrower with a variable-rate loan can convert the loan into a fixed-rate loan by entering into a swap as a fixed-rate payer. Alternatively, a borrower with a fixed-rate loan can convert the loan into a variable-rate loan by entering into swap as a fixed-rate receiver. The **net interest payment** made by the borrower is equal to the loan interest payment minus any proceeds received from the swap plus any payment made under the terms of the swap.

In the example below, Company XYZ has a loan with a variable interest rate and uses an interest rate swap to net to a fixed rate loan.

Example 17.13

Company XYZ has \$100,000,000 of floating rate debt on which it makes annual interest payments. Company XYZ is concerned that interest rates will increase, so it enters into an interest rate swap with \$100,000,000 notional and a swap rate of 5%. One year later, the floating interest rate is 7%.

a. Calculate the payment made to Company XYZ from the interest rate swap at the end of 1 year.

b. Calculate the net interest payment made by Company XYZ at the end of 1 year.

Solution

a. Company XYZ is paying a floating rate on its debt, so to protect itself from the possibility of rising interest rates, it enters the swap as a fixed-rate payer, which means that it receives floating-rate payments from the swap. Since the floating rate of 7% is higher than the fixed rate of 5%, Company XYZ receives:

$$100,000,000 \times (0.07 - 0.05) = \mathbf{2,000,000}$$

b. The net interest payment is the floating rate payment minus the amount received from the swap:

$$100,000,000(0.07) - 2,000,000 = \mathbf{5,000,000}$$

17.08 Questions

Question 17.01

You are given the annual effective spot rates shown in the table below:

Maturity (in years)	1	2	3	4	5
Spot Rate	3.0%	4.0%	5.0%	6.0%	7.0%

You enter into a 5-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

Calculate the fixed rate that you pay.

A 5.0% B 5.5% C 6.2% D 6.7% E 7.2%

Question 17.02

You are given the following one-year forward rates, deferred for t years:

t	0	1	2
One-Year Forward Rate	2%	3%	3.5%

You enter into a 3-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

Calculate the fixed rate that you pay.

A 2.8% B 3.0% C 3.2% D 3.4% E 3.5%

Question 17.03

You are given the prices of 3 zero-coupon bonds as a percentage of their redemption values.

Maturity (in years)	1	2	3
Price	98%	96%	93%

You enter into a 3-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

Calculate the fixed rate that you pay.

A 1.40% B 2.44% C 3.20% D 3.44% E 6.40%

Question 17.04

You are given the prices of 4 zero-coupon bonds as a percentage of their redemption values.

Maturity (in months)	3	6	9	12
Price	99%	97%	95%	94%

You enter into a 1-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 3-month LIBOR rate. The notional amount of the swap is 100,000, and the swap has quarterly payments.

Calculate the fixed rate that you pay, expressed as a quarterly effective interest rate.

A 1.34% B 1.44% C 1.56% D 2.00% E 2.06%

Question 17.05

You are given the prices of two par value bonds:

Maturity (in years)	Price	Coupon	Par
8	76,777.78	3%	100,000
8	58,200.91	0%	100,000

The first bond above makes annual coupon payments.

You enter into an 8-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

Calculate the fixed rate that you pay.

A 6.50% B 6.75% C 7.00% D 7.25% E 7.50%

Question 17.06

You are given the annual effective yields of two par value bonds:

Maturity (in years)	Yield	Coupon	Par
8	6.87%	3%	100,000
8	7.00%	0%	100,000

The first bond above makes annual coupon payments.

You enter into an 8-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

Calculate the fixed rate that you pay.

A 6.50% B 6.75% C 7.00% D 7.25% E 7.50%

Question 17.07

You are given the prices of three par value bonds that make annual coupon payments:

Maturity (in years)	Price	Coupon	Par
1	99,516.91	3%	100,000
2	94,802.83	2%	100,000
3	129,505.11	15%	100,000

You enter into a 3-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

Calculate the fixed rate that you pay.

A 4.28% B 4.29% C 4.30% D 4.31% E 4.32%

Question 17.08

You are given the annual effective spot rates shown in the table below:

Maturity (in years)	1	2	3	4
Spot Rate	3.0%	3.5%	4.0%	4.5%

You enter into a 4-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

Calculate the fixed rate that you pay.

A 3.54% B 3.75% C 4.00% D 4.24% E 4.44%

Question 17.09

You are given the annual effective spot rates shown in the table below:

Maturity (in years)	1	2	3	4
Spot Rate	3.0%	3.5%	4.0%	4.5%

You enter into a 4-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

At the end of 1 year, the 1-year LIBOR rate is 4.0%.

Calculate the net payment that you make at the end of 2 years.

A 354 B 375 C 400 D 424 E 445

Question 17.10

You are given the annual effective spot rates shown in the table below:

Maturity (in years)	1	2	3	4
Spot Rate	3.0%	3.5%	4.0%	4.5%

You enter into a 4-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year LIBOR rate. The notional amount of the swap is 100,000, and the swap has annual payments.

At the end of 1 year, the spot rates have changed. The new spot rates are shown in the table below:

Maturity (in years)	1	2	3
Spot Rate	4.0%	5.0%	6.0%

Calculate the present value to you of the remaining 3-year swap at the end of 1 year, immediately after the first interest rate swap payment is made.

A 4,000 B 5,200 C 7,600 D 9,750 E 14,800

Question 17.11

Company A entered into a 5-year interest rate swap two years ago as the receiver of a fixed rate of 5.5%. The notional amount of the swap is 100,000. The swap payments are made annually.

Now there are three years left on the swap agreement.

Today, the annual effective spot interest rates are as shown in the table below:

Maturity (in years)	1	2	3	4	5
Spot Rate	4.5%	5.0%	5.5%	6.0%	7.0%

Calculate the market value of the swap from Company A's perspective.

A −5,039 B −97 C 0 D 97 E 5,039

Question 17.12

Company X enters into a deferred interest rate swap with a level notional amount of 100,000. The swap will mature in seven years. Company X will pay a fixed interest rate and receive a floating interest rate during the last three years of the seven-year swap. The floating interest rate will be the one-year spot interest rate at the start of each year.

The settlement period is one year.

The current annual effective spot interest rates are shown in the table below:

Maturity (in years)	1	2	3	4	5	6	7
Spot Rate	3.0%	4.0%	4.5%	5.0%	5.5%	6.0%	6.5%

No swapping of interest rates will occur during the first four years.

Calculate the swap rate.

A 7.75% B 8.00% C 8.25% D 8.50% E 8.75%

Question 17.13

Five years ago, Company X entered into a deferred interest rate swap with a level notional amount of 100,000. At that time, the swap was a seven-year swap and Company X agreed to pay a fixed interest rate and receive a floating interest rate during the last three years of the seven-year swap. The floating interest rate is the one-year spot interest rate at the start of each year.

The settlement period is one year.

Five years ago, the annual effective spot interest rates were the rates shown in the table below:

Maturity (in years)	1	2	3	4	5	6	7
Spot Rate	3.0%	4.0%	4.5%	5.0%	5.5%	6.0%	6.5%

No swapping of interest rates occurred during the first four years.

The second payment from the swap will be made one year from now.

The current annual effective spot interest rates are shown in the table below:

Maturity (in years)	1	2	3	4	5
Spot Rate	2.0%	3.0%	3.5%	4.0%	5.5%

Calculate the net payment made by Company X in one year.

A 3,000 B 4,500 C 5,000 D 5,500 E 6,500

Question 17.14

Five years ago, Company X entered into a deferred interest rate swap with a level notional amount of 100,000. At that time, the swap was a seven-year swap and Company X agreed to pay a fixed interest rate and receive a floating interest rate during the last three years of the seven-year swap. The floating interest rate is the one-year spot interest rate at the start of each year.

The settlement period is one year.

Five years ago, the annual effective spot interest rates were the rates shown in the table below:

Maturity (in years)	1	2	3	4	5	6	7
Spot Rate	3.0%	4.0%	4.5%	5.0%	5.5%	6.0%	6.5%

No swapping of interest rates occurred during the first four years.

The second payment from the swap will be made one year from now.

The current annual effective spot interest rates are shown in the table below:

Maturity (in years)	1	2	3	4	5
Spot Rate	2.0%	3.0%	3.5%	4.0%	5.5%

Calculate the current value of the swap to Company X.

A –13,631 B –10,563 C –9,497 D –7,350 E –6,615

Question 17.15

One year ago, Company X entered into a deferred interest rate swap with a level notional amount of 100,000. At that time, the swap was a seven-year swap and Company X agreed to pay a fixed interest rate and receive a floating interest rate during the last three years of the seven-year swap. The floating interest rate is the one-year spot interest rate at the start of each year.

The settlement period is one year.

One year ago, the annual effective spot interest rates were the rates shown in the table below:

Maturity (in years)	1	2	3	4	5	6	7
Spot Rate	3.0%	4.0%	4.5%	5.0%	5.5%	6.0%	6.5%

No swapping of interest rates occurs during the first four years of the seven-year swap.

The first payment from the swap will be made in four years.

The current annual effective spot interest rates are shown in the table below:

Maturity (in years)	1	2	3	4	5	6
Spot Rate	2.0%	3.0%	3.5%	4.0%	5.5%	5.8%

Calculate the current value of the swap to Company X.

A –1,035 B –980 C –884 D –838 E –606

Question 17.16

You are given the annual effective spot rates shown in the table below:

Maturity (in years)	1	2	3
Spot Rate	2.50%	3.00%	4.19%

You enter into a 3-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year spot rate. The settlement period is one year.

The swap is an amortizing swap. The notional amount is 500,000 in the first year, 400,000 in the second year, and 300,000 in the third year.

Calculate the swap rate.

A 3.8% B 4.0% C 4.2% D 4.4% E 4.6%

Question 17.17

You are given the annual effective spot rates shown in the table below:

Maturity (in years)	1	2	3
Spot Rate	2.50%	3.00%	4.19%

You enter into a 3-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year spot rate. The settlement period is one year.

The swap is an amortizing swap. The notional amount is 500,000 in the first year, 400,000 in the second year, and 300,000 in the third year.

It is now one year later, and the spot rates have not changed, so the 1-year, 2-year, and 3-year spot rates remain 2.50%, 3.00%, and 4.19% respectively.

Calculate the net amount of the second swap payment that you make.

A −6,500 B −5,200 C 0 D 5,200 E 6,500

Question 17.18

You are given the annual effective spot rates shown in the table below:

Maturity (in years)	1	2	3
Spot Rate	2.50%	3.00%	4.19%

You enter into a 3-year interest rate swap in which you pay a fixed rate and receive a floating rate that is equal to the 1-year spot rate. The settlement period is one year.

The swap is an amortizing swap. The notional amount is 500,000 in the first year, 400,000 in the second year, and 300,000 in the third year.

It is now one year later, and the spot rates have not changed, so the 1-year, 2-year, and 3-year spot rates remain 2.50%, 3.00%, and 4.19% respectively.

Calculate the current value of the swap to you.

A −5,911 B −5,071 C −1,732 D −1,301 E 0

Question 17.19

The one-year forward rates, deferred for t years, are described in the table below:

t	0	1	2	3	4
One-Year Forward Rate	2.0%	2.5%	3.0%	4.0%	5.4%

You enter into a deferred interest rate swap that requires that you make fixed payments at the end of years 3, 4, and 5. The floating rate is the 1-year spot rate.

The swap is an accreting swap. The notional amount is 10,000,000 in the third year, 20,000,000 in the fourth year, and 30,000,000 in the fifth year.

Calculate the swap rate.

A 4.4% B 4.5% C 4.6% D 4.7% E 4.8%

Chapter 18: Banking

18.01 Depository Institutions

Banks, savings & loans, and credit unions are depository institutions that serve as **financial intermediaries** between borrowers with lenders. It would be time-consuming and cumbersome for lenders and borrowers to find each other and enter into loan agreements. Financial institutions such as banks are equipped to simplify and streamline the process. Lenders deposit funds with banks, and the banks make those funds available to borrowers.

The banking industry is evolving as technology makes internet banking a practical alternative for many of its customers. Internet-based banks are able to avoid significant overhead costs by maintaining few, if any, physical bank branches. Banks are able to pass the savings on to their customers in the form of higher interest rates for depositors and lower interest rates for borrowers. The key objective for banks remains, however, to charge borrowers more interest than they pay out to depositors.

Some of the factors that affect the interest rates encountered by bank customers include:

- Credit quality of the borrower
- Credit quality of the bank
- Type of loan
- Liquidity provided to the depositor
- Embedded options, such as the option to pay off a loan early
- Embedded guarantees, such as a guaranteed interest rate
- Overhead costs
- Profit requirements of the banking institutions
- Regional business environment
- Strategies of banking institutions

The banking industry faces challenges and disruption from alternatives that perform many of the functions traditionally performed only by banking institutions. Bitcoin payment networks, smart phone-based payment methods, and other e-commerce payment systems are part of an emerging financial technology, known as **fintech**, that is changing the nature of financial transactions.

Banking activities that take place outside of the regulated banking sector are called **shadow banking**. Shadow banking institutions are able to sidestep many of the regulations on the banking industry by performing only some of functions performed by traditional banks. For example, some lending institutions do not accept deposits, instead raising their lendable funds from investors. To the extent that an economy relies on shadow banking institutions remaining functional and solvent, they can jeopardize the economy if they fail during times of economic distress. Several shadow banking institutions failed during the financial crisis of 2007−2009, further destabilizing the economy at a difficult time.

Interest Rates on Deposits

Banking institutions accept deposits primarily into three types of accounts:

- Checking accounts usually do not pay interest, but the funds are immediately available.
- Money market and savings accounts pay interest and allow immediate access to the funds, although the access is sometimes not quite as convenient as with checking accounts.
- Certificates of deposit usually pay higher interest than money market and savings accounts, but access to the funds is limited by early withdrawal penalties.

The financial strength of a bank sometimes plays a role in determining the interest rate it pays on deposits. A bank with a relatively low credit rating may offer higher interest rates to its depositors to compensate for the risk that the bank could fail. The Federal Deposit Insurance Corporation (FDIC) insures bank deposits, but a bank failure can still result in delayed access to deposits. Furthermore, FDIC insurance is capped at $250,000 per depositor per bank.

Regional differences in local economies can cause credited rates to vary across the country. The business environment in a region may lead to higher or lower interest rates than the rates observed elsewhere. For example, in a part of the country where savings deposits are unusually high, interest rates may be lower than in other areas.

A bank's business strategy can affect its crediting rate. Banks that are seeking high growth may pay higher interest rates to attract additional deposits. Banks that focus on financial stability may charge higher rates for risky loans than the rates charged by their competitors.

Interest Rates on Loans

The interest rates charged by a bank are higher when the probability of default losses is higher. The three primary types of loans issued by a bank are the following:

- Unsecured loans do not require the borrower to pledge collateral to obtain the loan. Credit card debt is typically unsecured.
- Secured loans require the borrower to pledge that an asset, such as a house or car, will be turned over to the lender in the event that the borrower fails to make the loan payments. The pledged asset is often referred to as collateral. Home and auto loans are common examples of secured loans.
- Guaranteed loans are guaranteed by a third party, called the guarantor, which agrees to make the loan payments in the event that the borrower fails to do so. An example of a loan with guaranteed repayment is a loan for which a parent or grandparent has agreed to serve as the guarantor. Another example, is a student loan for which the government serves as the guarantor.

Unsecured loans tend to have the highest interest rates and guaranteed loans tend to have the lowest rates. However, the interest rate on a secured loan can be lower than the rate on a guaranteed loan if the secured loan is secured with sufficiently valuable collateral, or the guarantor of the guaranteed loan has a relatively poor credit rating.

The borrower's credit rating affects the interest rate that the borrower must pay. Borrowers with consistent income and a record of making loan payments on time usually receive lower interest rates than higher risk borrowers. Borrowers with good credit histories often find that they are able to negotiate mortgage rates that are less than the stated mortgage rates. One reason for a bank to provide a reduced mortgage rate is that the bank may earn fees and income on other banking services utilized by the borrower.

Regulations on credit cards, however, require banks to charge the same interest rate to all of the cardholders of the same credit card product. Banks are sometimes able to get around this requirement by tailoring their credit cards to target customers within specified bands of creditworthiness. For example, cards with relatively low income-qualification requirements may have higher interest rates.

A bank's **prime rate** is the interest rate charged for loans to its best customers. The average of the prime rates charged by the largest banks is often called simply *the* prime rate. Large banks nearly always set their prime rate to be 300 basis points above the federal funds target rate (which is discussed in the next section), so in recent years the prime rate has followed the following formula:

$$\text{Prime rate} = \text{Fed funds target} + 0.03$$

The prime rate is considered an important indicator of the general level of interest rates, and many floating-rate loans have interest rates that are tied to the prime rate.

18.02 Central Banks

A central bank is a governmental entity that serves as a bank for a country's depository institutions, such as its banks, savings & loans, and credit unions.

In the United States the central bank is known as the Federal Reserve System, which is often shortened to the Federal Reserve or simply the Fed. The United States is divided into 12 Federal Reserve Districts, each of which has its own Federal Reserve Bank. Since the Federal Reserve is a system of 12 banks, it is called the Federal Reserve System.

The Federal Reserve System has three decision-making entities:

- a 7-member Board of Governors, which is appointed by the President of the U.S.
- the Federal Open Market Committee (FOMC), which is described later in this section
- the 12 Federal Reserve Banks

The functions of a central bank vary by country, but all central banks have the following two responsibilities:

3. Maintain the safety and soundness of the nation's banking system
4. Support the smooth operation of the nation's payment and settlement system

In addition, many central banks are responsible for conducting and implementing monetary policy. Each of these functions is discussed below.

Banking System Safety

Banks maintain deposits with the central bank, and these deposits are known as reserves. Each bank has a **reserve requirement**, which is the minimum required amount that the bank must hold with the central bank. The reserve requirement fluctuates from day to day, largely based on the total amount of deposits at the bank. The reserve requirement is also affected by the volume of transactions conducted by the bank and the relative significance of the bank within the national banking system.

If a bank does not have enough reserves on deposit with the Federal Reserve, then the bank can borrow reserves from the Federal Reserve at an interest rate that is referred to as the **discount rate**. It is considered a sign of financial weakness to borrow from the Federal Reserve though, so it is more common for banks to borrow reserves from each other. Banks with excess reserves provide overnight lending to banks with insufficient reserve levels. In the United States, the interest rate paid by one bank to another for the use of the reserves is called the **federal funds rate**. The federal funds rate is usually less than the discount rate, which is another reason for banks to borrow reserves from each other instead of from the Federal Reserve.

If a bank cannot persuade other banks to lend to it, then the bank is forced to turn to the central bank to obtain reserves. Since the central bank always stands ready to lend to a bank, the central bank is sometimes called the lender of last resort. The central bank is able to provide such loans because it has two sources of funds. First, it has the reserves deposited by other banks. Second, many central banks are able to unilaterally increase a bank's reserves, thereby creating money. The Federal Reserve has this ability to create money, and this allows it to accommodate large withdrawals from troubled banks. If the Federal Reserve did not have this ability to create money, it would likely to be necessary to limit the withdrawals from troubled banks.

In the absence of a central bank, banks could find themselves subject to a run on the bank. A run on the bank occurs when a bank's customers become concerned about the financial strength of a bank and therefore rush to withdraw their funds. Banks don't hold all of their depositors funds in cash because they use those funds to make loans. Consequently, a run on the bank can quickly exhaust a bank's cash reserves. Having a central bank serve as the lender of last resort allows the bank to satisfy its customers' withdrawal requests. Furthermore, awareness of the existence of the central bank and its ability to support the banking institution make a run on the bank unlikely in the first place.

Operation of the Payment and Settlement System

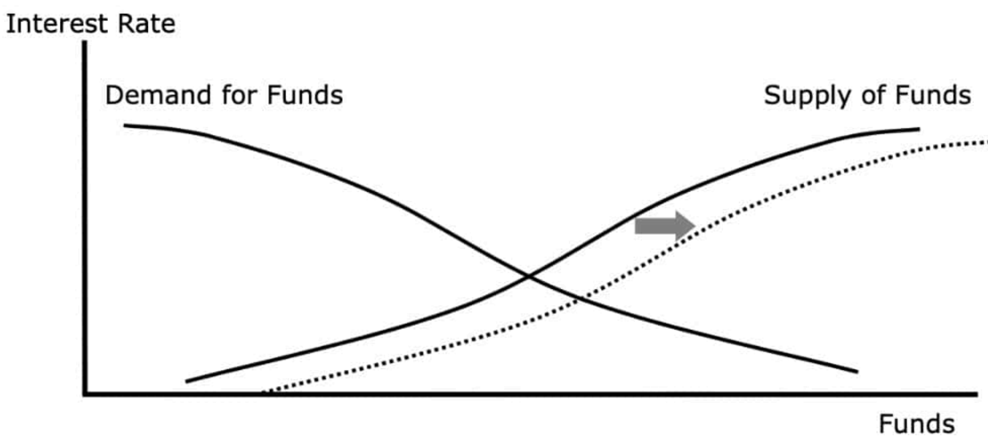
Central banks play an important role in facilitating the payments made for goods and services throughout the nation. For example, suppose that John wishes to purchase apples from Jane. John can write a check to Jane, and Jane can deposit the check into her bank account. If John and Jane use different banks, then John's bank needs to transfer funds to Jane's bank. John's bank can accomplish this by instructing the central bank to transfer some of its reserves to Jane's bank. By transferring reserves from one bank to another, the central bank allows depository institutions to easily and efficiently transfer funds between one other.

Monetary Policy

A country's **monetary policy** is its plan for the management of the country's money supply. The Federal Reserve conducts the monetary policy of the United States and strives to achieve three, sometimes contradictory, goals: maximum employment levels, stable prices, and moderate long-term interest rates.

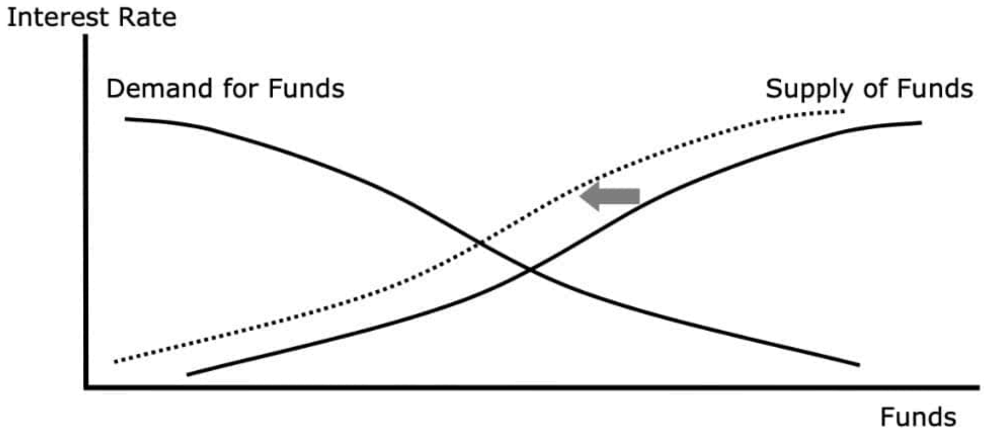
The monetary policy is set by the Federal Open Market Committee (FOMC), which is part of the Federal Reserve System. The FOMC has 12 members: the seven members of the Board of Governors, the president of the Federal Reserve Bank of New York, and four of the eleven remaining Federal Reserve district bank presidents. The FOMC uses a target for the federal funds rate to signal its intentions. A higher federal funds target could indicate that the FOMC is concerned about inflation and is therefore planning to reduce the money supply. A lower federal funds target could indicate that the FOMC believes the economy needs some stimulus in the form of lower interest rates, perhaps with the goal of increasing employment levels. The FOMC implements its monetary policy through open market operations, which consist of the purchase and sale of securities issued or backed by the U.S. government.

When the FOMC purchases government securities, it pays for those securities by crediting (increasing) the reserve accounts of the securities dealers' banks. This increase in reserves reduces the federal funds rate and increases the funds available for banks to lend. As shown below, the supply curve for money is shifted to the right, and the interest rate decreases.



A lower federal funds rate makes it less expensive for banks to have a shortfall in their reserve accounts because the cost of borrowing to cover the shortfall is lower. Therefore, in response to a lower federal funds rate, banks write more loans, and the interest rates on those loans decrease.

When the FOMC sells government securities, the reserve accounts of the securities dealers' banks are reduced. This decrease in reserves increases the federal funds rate and decreases the funds available for banks to lend. As shown below, the supply curve is shifted to the left, and the interest rate increases.



A higher federal funds rate makes it more expensive for banks to have a shortfall in their reserve accounts because the cost of borrowing to cover the shortfall is higher. Therefore, in response to a higher federal funds rate, banks write fewer loans, and the interest rates on those loans increase.

18.03 Questions

Question 18.01

Which of the following statements is FALSE?

A The region in which a bank operates may affect the interest that the bank pays on savings products.

B Home mortgages, auto loans, and home equity loans are examples of unsecured loans.

C A bank's prime rate is the rate it charges its most creditworthy customers.

D The mortgage rates posted by banks are typically higher than what a customer with a good credit history can expect to pay.

E A mortgage that is guaranteed by a mortgage insurance company is a guaranteed loan.

Question 18.02

Which of the following statements is FALSE?

A The Federal Open Market Committee is responsible for setting monetary policy.

B A central bank acts as a banker to the commercial banks in its country.

C The amount of money that a bank must maintain on deposit at the central bank is known as the bank's reserve requirement.

D The reserve accounts that banks maintain at the central bank provide an efficient way of moving money from one bank to another.

E The federal funds rate is the rate at which banks can borrow directly from the Federal Reserve.

Question 18.03

You are given the following interest rates:

- The discount rate is 2.5%.
- The federal funds rate is 1.9%.
- The federal funds target rate is 2.0%.
- The prime rate is 5.0%.
- The LIBOR is 2.1%.

A bank has excess reserves on deposit with the Federal Reserve and lends the excess deposits to another bank. What is the rate at which the excess reserves are lent?

A 1.9% B 2.0% C 2.1% D 2.5% E 5.0%

Question 18.04

The Federal Open Market Committee has announced a new, higher target level for the federal funds rate.

Determine which of the following best describes the impact on the banking sector and interest rates.

A It becomes more expensive for banks to run a shortfall in their reserve accounts, so banks will write fewer loans, and the interest rates charged on those loans will decrease.

B It becomes more expensive for banks to run a shortfall in their reserve accounts, so banks will write fewer loans, and the interest rates charged on those loans will increase.

C It becomes more expensive for banks to run a shortfall in their reserve accounts, so banks will write more loans, and the interest rates charged on those loans will decrease.

D It becomes less expensive for banks to run a shortfall in their reserve accounts, so banks will write more loans, and the interest rates charged on those loans will increase.

E It becomes less expensive for banks to run a shortfall in their reserve accounts, so banks will write more loans, and the interest rates charged on those loans will decrease.

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Answer Key

Question	Answer	Topic	Section	Difficulty
1.01	C	Terminology	1.04	1
1.02	E	Supply and Demand for Funds	1.05	1
2.01	D	Simple Interest	2.01	1
2.02	D	Simple Interest	2.01	1
2.03	A	Simple Discount	2.02	1
2.04	B	Simple Discount	2.02	2
2.05	E	Simple Interest & Discount	2.02	2
2.06	C	Simple Interest & Discount	2.03	2
2.07	A	Simple Interest & Discount	2.03	1
2.08	B	Simple Interest	2.04	1
2.09	A	Simple Interest & Discount	2.04	3
2.10	C	Simple Interest & Discount	2.04	5
3.01	D	Compound Interest	3.01	1
3.02	D	Compound Interest	3.01	2
3.03	C	Compound Interest	3.01	1
3.04	E	Compound Interest	3.01	2
3.05	B	Compound Interest	3.01	2
3.06	E	Simple Interest & Compound Interest	3.01	1
3.07	A	Compound Interest	3.01	2
3.08	C	Compound Interest	3.01	3
3.09	A	Compound Interest	3.01	3
3.10	B	Compound Interest	3.01	3
3.11	C	Compound Discount	3.03	1
3.12	E	Compound Discount	3.03	1
3.13	B	Compound Interest and Discount	3.03	1
3.14	B	Compound Discount	3.03	1
3.15	B	Compound Discount	3.03	3
3.16	E	Equivalent Compound Interest and Discount	3.04	2
3.17	D	Interest Rate Conversions	3.05	1
3.18	E	Interest Rate Conversions	3.05	2
3.19	A	Simple Interest & Compound Interest	3.05	2
3.20	C	Interest Rate Conversions	3.05	1
3.21	D	Nominal Interest Rates	3.05	2
3.22	D	Nominal Interest Rates	3.05	3
3.23	E	Nominal Interest Rates	3.05	3
3.24	D	Nominal Interest Rates	3.05	4
3.25	B	Nominal Interest Rates	3.05	4
3.26	D	Nominal Interest and Discount Rates	3.06	2
3.27	D	Nominal Discount Rates	3.06	4
3.28	C	Interest Rate Conversions	3.07	1

Question	Answer	Topic	Section	Difficulty
3.29	C	Interest Rate Conversions	3.07	1
3.30	B	Nominal Discount Rates	3.07	4
4.01	B	Force of Interest	4.02	1
4.02	A	Force of Interest	4.02	1
4.03	C	Force of Interest	4.02	1
4.04	B	Force of Interest	4.02	1
4.05	A	Force of Interest	4.02	1
4.06	N/A	Force of Interest	4.02	1
4.07	B	Force of Interest	4.02	1
4.08	D	Force of Interest	4.02	1
4.09	E	Force of Interest	4.02	2
4.10	A	Force of Interest	4.02	2
4.11	C	Force of Interest	4.02	2
4.12	E	Force of Interest	4.02	2
4.13	E	Force of Interest	4.02	2
4.14	A	Force of Interest	4.02	2
4.15	D	Force of Interest	4.02	4
5.01	A	Varying Compound Interest Rates	5.01	2
5.02	C	Varying Compound Interest Rates	5.01	2
5.03	B	Varying Compound Discount Rates	5.02	2
5.04	B	Varying Compound Discount Rates	5.02	2
5.05	B	Varying Compound Discount Rates	5.02	2
5.06	A	Varying Force of Interest	5.03	1
5.07	D	Varying Force of Interest	5.03	2
5.08	N/A	Varying Force of Interest	5.03	2
5.09	N/A	Varying Force of Interest	5.03	2
5.10	C	Varying Force of Interest	5.03	3
5.11	B	Varying Force of Interest	5.03	3
5.12	A	Varying Force of Interest	5.03	3
5.13	D	Varying Force of Interest	5.03	4
5.14	C	Varying Force of Interest	5.03	4
5.15	E	Varying Force of Interest	5.03	4
5.16	C	Varying Force of Interest	5.03	4
5.17	D	Varying Force of Interest	5.03	4
5.18	B	Varying Force of Interest	5.03	4
5.19	D	Varying Force of Interest	5.03	5
5.20	E	Varying Force of Interest	5.03	5
5.21	E	Mix of Varying Rates	5.04	1
5.22	A	Mix of Varying Rates	5.04	1
5.23	C	Mix of Varying Rates	5.04	2
5.24	C	Mix of Varying Rates	5.04	2
5.25	C	Accumulation Function	5.05	1
6.01	E	Annuity-Immediate	6.02	1

Question	Answer	Topic	Section	Difficulty
6.02	B	Annuity Immediate	6.02	2
6.03	A	Annuity-Immediate	6.02	2
6.04	D	Annuity-Immediate	6.02	2
6.05	B	Annuity-Immediate	6.02	2
6.06	C	Annuity-Immediate	6.02	2
6.07	B	Annuity-Immediate	6.02	3
6.08	C	Annuity-Immediate	6.02	3
6.09	B	Annuity-Immediate	6.02	3
6.10	B	Annuity-Immediate	6.02	3
6.11	C	Annuity-Immediate	6.02	3
6.12	A	Annuity-Immediate	6.02	3
6.13	C	Annuity-Due	6.03	2
6.14	C	Annuity-Due	6.03	2
6.15	B	Annuity-Due	6.03	2
6.16	D	Annuity-Due	6.03	2
6.17	D	Annuity-Immediate & Annuity-Due	6.03	2
6.18	C	Annuity-Immediate & Annuity-Due	6.03	2
6.19	D	Annuity-Immediate & Annuity-Due	6.03	3
6.20	A	Annuity-Due	6.03	3
6.21	D	Annuity-Due	6.03	3
6.22	E	Annuity-Due	6.03	3
6.23	D	Annuity-Due	6.03	4
6.24	B	Annuity-Due	6.03	4
6.25	B	Level Annuities	6.03	5
6.26	B	Deferred Annuities	6.04	1
6.27	D	Deferred Perpetuities	6.04	2
6.28	C	Deferred Annuities	6.04	2
6.29	B	Deferred Annuities	6.04	3
6.30	E	Deferred Perpetuities	6.04	4
7.01	D	Annuity-Immediate	7.01	1
7.02	D	Perpetuity-Immediate	7.01	1
7.03	A	Annuity-Immediate	7.01	2
7.04	D	Annuity-Immediate	7.01	3
7.05	C	Annuity-Immediate	7.01	3
7.06	E	Annuity-Immediate	7.01	3
7.07	B	Perpetuity-Immediate	7.01	3
7.08	B	Annuity-Immediate	7.01	4
7.09	E	Annuity-Immediate	7.01	5
7.10	E	Annuity-Due	7.02	1
7.11	C	Perpetuity-Due	7.02	1
7.12	E	Annuity-Due	7.02	2
7.13	D	Annuity-Due	7.02	2
7.14	B	Annuity-Due	7.02	3

Question	Answer	Topic	Section	Difficulty
7.15	A	Annuity-Due	7.02	3
7.16	B	Annuity-Due	7.02	3
7.17	B	Perpetuity-Due	7.02	4
7.18	C	Level Annuities	7.02	4
7.19	D	Annuity-Due	7.02	4
7.20	D	Level Annuities, Payable Continuously	7.03	1
7.21	E	Level Annuities, Payable Continuously	7.03	2
7.22	D	Level Annuities, Payable Continuously	7.03	4
7.23	A	Level Annuities	7.03	4
8.01	D	Increasing Annuity	8.01	2
8.02	C	Increasing Annuity	8.01	2
8.03	D	Increasing Annuity	8.01	2
8.04	E	Annuity Increasing once per time unit	8.01	3
8.05	D	Annuity Increasing once per time unit	8.01	3
8.06	A	Two Interest Rates	8.01	4
8.07	C	Two Interest Rates	8.01	4
8.08	B	Increasing Coupons	8.01	5
8.09	C	Increasing Annuity	8.02	2
8.10	B	Varying Annuity	8.02	3
8.11	D	Increasing Annuity	8.02	3
8.12	C	Increasing Annuity	8.02	3
8.13	E	Increasing Annuity	8.02	3
8.14	B	Annuity Increasing mthly, payable pthly	8.02	3
8.15	E	Continuous Annuity, Increasing Once per Time Unit	8.02	3
8.16	C	Annuity Increasing mthly, payable pthly	8.02	4
8.17	E	General Form of Arithmetic Progression Annuity	8.02	4
8.18	A	Increasing Perpetuity	8.03	2
8.19	D	Increasing Perpetuity	8.03	2
8.20	E	Increasing Perpetuity	8.03	2
8.21	B	Varying Annuity	8.03	4
8.22	D	Decreasing Annuity	8.04	2
8.23	E	Decreasing Annuity	8.04	2
8.24	C	Decreasing Annuity	8.04	2
8.25	A	Decreasing Annuity	8.04	3
8.26	E	Decreasing Annuity	8.04	3
8.27	E	Annuity Decreasing once per time unit	8.04	3
8.28	D	Annuity Decreasing mthly, payable pthly	8.04	3
8.29	E	Continuous Annuity, Decreasing Once per Time Unit	8.04	3
8.30	C	Decreasing Annuity	8.04	4
8.31	C	Decreasing Annuity	8.04	4
9.01	B	Continuously Varying Payment Stream	9.02	2
9.02	E	Continuously Varying Payment Stream	9.01	3
9.03	D	Continuous Annuity, Increasing Continuously	9.02	3

Question	Answer	Topic	Section	Difficulty
9.04	D	Continuous Annuity, Increasing Continuously	9.02	3
9.05	A	Continuous Annuity, Decreasing Continuously	9.02	3
9.06	C	Continuously Varying Payment Stream	9.01	4
9.07	A	Continuously Varying Payment Stream	9.01	4
9.08	C	Continuously Varying Payment Stream	9.01	5
9.09	B	Continuously Varying Payment Stream	9.02	5
9.10	A	Continuously Varying Payment Stream	9.01	4
10.01	D	Geometric Progression Annuity	10.02	1
10.02	A	Geometric Progression Annuity	10.02	2
10.03	D	Geometric Progression Annuity	10.02	2
10.04	A	Geometric Progression Annuity	10.02	2
10.05	B	Geometric Progression Annuity	10.02	3
10.06	E	Geometric Progression Annuity	10.02	3
10.07	C	Geometric Progression Annuity	10.02	3
10.08	B	Geometric Progression Annuity	10.02	4
10.09	B	Geometric Progression Annuity	10.02	4
10.10	A	Geometric Progression Annuity	10.02	4
10.11	C	Geometric Progression Annuity	10.02	4
10.12	E	Geometric Progression Annuity	10.02	4
10.13	D	Geometric Progression Annuity	10.02	4
10.14	A	Geometric Progression Annuity	10.02	5
10.15	C	Geometric Progression Annuity	10.02	5
10.16	E	Dividend Discount Model	10.03	1
10.17	E	Dividend Discount Model	10.03	1
10.18	B	Dividend Discount Model	10.03	2
10.19	C	Dividend Discount Model	10.03	3
10.20	B	Dividend Discount Model	10.03	3
11.01	A	Level Payment Amortized Loans	11.02	2
11.02	A	Level Payment Amortized Loans	11.02	2
11.03	E	Level Payment Amortized Loans	11.02	2
11.04	E	Level Payment Amortized Loans	11.02	2
11.05	B	Level Payment Amortized Loans	11.02	3
11.06	E	Level Payment Amortized Loans	11.02	3
11.07	C	Level Payment Amortized Loans	11.02	2
11.08	D	Level Payment Amortized Loans	11.02	2
11.09	C	Level Payment Amortized Loans	11.02	3
11.10	B	Level Payment Amortized Loans	11.02	3
11.11	D	Level Payment Amortized Loans	11.02	3
11.12	D	Level Payment Amortized Loans	11.02	4
11.13	C	Level Payment Amortized Loans	11.02	4
11.14	B	Level Payment Amortized Loans	11.02	4
11.15	E	Level Payment Amortized Loans	11.02	5
11.16	E	Level Payment Amortized Loans	11.02	5

Question	Answer	Topic	Section	Difficulty
11.17	C	Level Payment Amortized Loans	11.02	5
11.18	A	Drop Payments	11.03	3
11.19	D	Drop Payments	11.03	4
11.20	B	Sinking Fund	11.04	2
11.21	A	Sinking Fund	11.04	2
11.22	A	Sinking Fund	11.04	3
11.23	B	Sinking Fund	11.04	3
11.24	C	Sinking Fund	11.04	3
12.01	B	Net Present Value	12.01	1
12.02	D	Net Present Value	12.01	2
12.03	C	Internal Rate of Return	12.02	1
12.04	B	Internal Rate of Return	12.02	1
12.05	E	Internal Rate of Return	12.02	2
12.06	A	Dollar-Weighted Rate of Return	12.02	3
12.07	A	Time-Weighted Rate of Return	12.03	1
12.08	D	Time-Weighted Rate of Return	12.03	2
12.09	A	Time-Weighted & Dollar-Weighted Returns	12.03	3
12.10	E	Time-Weighted & Dollar-Weighted Returns	12.03	3
12.11	C	Time-Weighted & Dollar-Weighted Returns	12.03	4
12.12	C	Defaults	12.04	2
12.13	A	Defaults	12.04	2
12.14	D	Inflation	12.05	1
12.15	D	Inflation	12.05	1
12.16	B	Inflation	12.05	1
12.17	E	Inflation	12.05	1
12.18	E	Inflation	12.05	2
12.19	E	Inflation	12.05	4
12.20	B	Inflation	12.05	4
13.01	C	Pricing Noncallable Bonds	13.01	1
13.02	D	Pricing Noncallable Bonds	13.01	1
13.03	B	Pricing Noncallable Bonds	13.01	1
13.04	D	Pricing Noncallable Bonds	13.01	1
13.05	E	Pricing Noncallable Bonds	13.01	2
13.06	A	Pricing Noncallable Bonds	13.01	2
13.07	C	Pricing Noncallable Bonds	13.01	2
13.08	E	Pricing Noncallable Bonds	13.01	2
13.09	A	Pricing Noncallable Bonds	13.01	2
13.10	B	Noncallable Bonds	13.01	3
13.11	D	Pricing Noncallable Bonds	13.01	3
13.12	B	Noncallable Bonds	13.01	3
13.13	C	Noncallable Bonds	13.01	3
13.14	E	Noncallable Bonds	13.01	4
13.15	A	Pricing Noncallable Bonds	13.01	5

Question	Answer	Topic	Section	Difficulty
13.16	E	Noncallable Bonds	13.02	2
13.17	B	Bond Investment Income	13.02	2
13.18	A	Bond Book Values	13.02	3
13.19	D	Bond Book Values	13.02	4
13.20	C	Bond Book Values	13.02	4
13.21	A	Callable Bonds	13.03	2
13.22	D	Callable Bonds	13.03	2
13.23	B	Callable Bonds	13.03	2
13.24	C	Callable Bonds	13.03	3
13.25	D	Callable Bonds	13.03	3
13.26	B	Callable Bonds	13.03	3
13.27	C	Callable Bonds	13.03	3
13.28	E	Callable Bonds	13.03	3
13.29	A	Government Bonds	13.04	1
13.30	B	Government Bonds	13.04	1
13.31	A	Corporate Bonds	13.05	2
13.32	D	U.S. T-Bills	13.06	1
13.33	E	U.S. T-Bills	13.06	3
13.34	B	Canadian T-Bills	13.07	1
13.35	C	Canadian T-Bills	13.07	3
13.36	A	T-Bills	13.07	3
13.37	E	T-Bills	13.07	3
14.01	D	Macaulay Duration	14.01	1
14.02	A	Portfolio Duration	14.01	1
14.03	E	Macaulay Duration	14.01	1
14.04	C	Macaulay Duration	14.01	1
14.05	D	Macaulay Duration	14.01	2
14.06	C	Macaulay Duration	14.01	2
14.07	D	Macaulay Duration	14.01	2
14.08	D	Macaulay Duration	14.01	3
14.09	B	Macaulay Duration	14.01	4
14.10	B	Macaulay Duration	14.01	4
14.11	C	Macaulay Duration	14.01	4
14.12	B	Macaulay Duration	14.01	5
14.13	E	Modified Duration	14.02	1
14.14	D	Modified Duration	14.02	2
14.15	A	Modified Duration	14.02	2
14.16	E	Modified Duration	14.02	3
14.17	C	Modified Duration	14.02	3
14.18	A	Modified Duration	14.02	3
14.19	D	Modified Duration	14.02	4
14.20	D	Convexity	14.03	2
14.21	B	Modified Convexity	14.03	2

Question	Answer	Topic	Section	Difficulty
14.22	D	Macaulay Convexity	14.03	2
14.23	C	Modified Convexity	14.03	3
14.24	E	Convexity	14.03	3
14.25	C	Macaulay Convexity	14.03	4
14.26	D	FoMac Approximation	14.05	1
14.27	C	FoMac Approximation	14.05	1
14.28	A	FoMac Approximation	14.05	2
14.29	D	FoMac Approximation	14.05	2
14.30	B	FoMac Approximation	14.05	2
15.01	B	Dedication	15.02	1
15.02	C	Dedication	15.02	1
15.03	B	Dedication	15.02	1
15.04	B	Dedication	15.02	1
15.05	B	Dedication	15.02	2
15.06	A	Dedication	15.02	2
15.07	D	Asset-Liability Management	15.02	3
15.08	B	Redington Immunization	15.03	2
15.09	E	Redington Immunization	15.03	2
15.10	E	Immunization	15.03	2
15.11	C	Redington Immunization	15.03	2
15.12	C	Interest Rate Risk	15.01	3
15.13	A	Redington Immunization	15.03	3
15.14	D	Redington Immunization	15.03	4
15.15	A	Redington Immunization	15.03	4
15.16	A	Asset-Liability Management	15.04	1
15.17	D	Full Immunization	15.04	2
15.18	C	Full Immunization	15.04	3
15.19	C	Full Immunization	15.04	4
16.01	C	Spot Rates	16.02	1
16.02	E	Spot Rates	16.02	1
16.03	C	Spot Rates	16.02	2
16.04	B	Spot Rates	16.02	3
16.05	B	Forward Rates	16.03	1
16.06	D	Forward Rates	16.03	1
16.07	E	Forward Rates	16.03	1
16.08	E	Forward Rates	16.03	2
16.09	D	Forward Rates	16.03	3
16.10	B	Forward Rates	16.03	1
16.11	A	Forward Rates	16.03	2
16.12	C	Forward Rates	16.03	2
16.13	C	Spot Rates	16.02	2
16.14	A	Forward Rates	16.03	2
16.15	D	Yield Curve Theories	16.05	1

Question	Answer	Topic	Section	Difficulty
17.01	D	Establishing the Swap Rate	17.01	2
17.02	A	Establishing the Swap Rate	17.01	2
17.03	B	Establishing the Swap Rate	17.01	2
17.04	C	Establishing the Swap Rate	17.01	3
17.05	B	Establishing the Swap Rate	17.01	3
17.06	B	Establishing the Swap Rate	17.01	4
17.07	C	Establishing the Swap Rate	17.01	4
17.08	E	Establishing the Swap Rate	17.01	2
17.09	E	Interest Rate Swap Payment	17.01	2
17.10	A	Valuing Interest Rate Swaps	17.02	3
17.11	D	Valuing Interest Rate Swaps	17.02	2
17.12	D	Deferred Interest Rate Swaps	17.04	2
17.13	E	Deferred Interest Rate Swaps	17.04	3
17.14	B	Deferred Interest Rate Swaps	17.04	3
17.15	C	Deferred Interest Rate Swaps	17.04	4
17.16	A	Amortizing Swaps	17.05	3
17.17	D	Amortizing Swaps	17.05	3
17.18	A	Amortizing Swaps	17.05	4
17.19	B	Varying and Accreting Swap	17.05	4
18.01	B	Depository Institutions	18.01	1
18.02	E	Central Banks	18.02	1
18.03	A	Central Banks	18.02	1
18.04	B	Central Banks	18.02	1

Index

G

H

I

J

L

M

N

O

P

Q

R

S

T

U

V

Y

Z